

Università di Torino

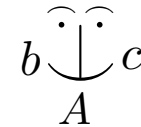
QUADERNI DIDATTICI

del

Dipartimento di Matematica

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A crèche course in model theory



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Contents

Readme	3
1 Structures and morphisms	5
1.1 Structures and isomorphism	5
1.2 Examples	10
2 First-order formulas	13
2.1 The syntax	13
2.2 The semantic	17
2.3 Definable sets	20
3 Elementarity	23
3.1 Theories and elementarity	23
3.2 Elementary maps and partial isomorphisms	28
3.3 The downward Löwenheim-Skolem theorem	32
4 Compactness	35
4.1 Consistency	35
4.2 Compactness	37
4.3 Monster models	40
4.4 Daily life in a saturated universe	44
5 Collapse of quantifiers	47
5.1 Elimination of quantifiers	47
5.2 Model-completeness	52
5.3 Examples	54
6 Saturation and homogeneity	61
6.1 Saturation	62
6.2 Homogeneity	64

7 Omitting elements	73
7.1 Omitting small sets	73
7.2 Omitting definable sets	79
7.3 Omitting type-definable sets	81
8 Poor man's geometry	85
8.1 Strongly minimal theories	85
8.2 Independence	91
8.3 Modularity and linearity	92
9 Prime and atomic models	97
9.1 Prime and atomic models	97
9.2 Uniqueness of strictly-prime models	103
10 Countable models	107
10.1 Countable atomic models	107
10.2 Countable categoricity	110
10.3 Finite axiomatizability	112
10.4 On the number of countable models	113
11 Generic structures	117
11.1 Generic structures	117
11.2 Saturation and quantifier elimination	121
11.3 Amalgamation classes	122
11.4 Examples	127
12 Definability and almost-definability	129
12.1 Definability and fixed points	129
12.2 Finite orbits of sets	131
12.3 Finite equivalence relations	133
13 Elimination of imaginaries	139
13.1 Definable sets as parameters	139
13.2 Elimination of imaginaries	141
13.3 The eq-theory	147
Bibliography	151
Index	153

Readme

These are the lecture notes of a PhD course given in 2004 at the University of Torino. They are based on notes used between 1998 and 2000 in the Master of Logic program of the University of Amsterdam.

For some time I have been seriously considering the possibility of writing a “real book”. Soon I realized what a presumptuous idea this was. Still, the many “undefined references” in these notes prove that I did not give up hope yet: when I grow up I want to write that book.

Description.

The course starts at undergraduate level with drills on saturation and homogeneity. It only appeals to the general interest in the interaction between syntax and semantics. Then it slowly moves to themes that are specific domain of model theory (but never go very deep).

Many students come from philosophy or from theoretical computer science and do not have a strong mathematical background. So the notions of algebra I use are always very basic and never essential. For instance I hardly consider any interaction of model theory with algebra and geometry. The interested reader is referred to the literature, a few pointers are included at the end.

Use.

These notes contain a few repetitions and as many references (back and forward) as I was able to insert. This should help the reader that wants to zig-zag through the various sections. I myself recommend Section 4.4 as the best starting point for people with some basic knowledge in logic.

Exercises.

Exercises are assigned all along the notes. They are integral part of the course and we refer to them in some proofs. Exercises range from completely trivial to mildly challenging. Paragraphs that contain an exercise are marked as this one. **EXERCISE**

Disclaimer.

I am indebted to all colleagues and students that helped me in finding errors in earlier versions. But these notes are still **under construction**, so new errors sneak in as others get out. So read them at your own risk.

Domenico Zambella
Torino, 24 March 2004

Chapter 1

Structures and morphisms

We define structures and introduce the concept of isomorphisms between structures. We also fix some basic notation.

1.1 Structures and isomorphism

Roughly, a structure consists of a set M , the domain of the structure, together with a collection of relations and functions on M of arbitrary arity. But this is not all: it is convenient to assume that these relations and functions are *labeled* in some way. In fact, a labeling enforces unambiguously a natural way to compare structures. This is common in mathematics, for instance, rings have two binary functions: one is labeled by $+$ and the other by \cdot . The set of labels for the functions and relations of a structure is called the signature (or the language) of the structure.

1.1.1 Language.

A **(first-order) signature** or **language** consists of:

- 1 a set L which is the disjoint union of two sets L_{function} and L_{relation} whose elements we call (symbols for) **relations** and, respectively, **functions**; and
- 2 a map $\text{Arity} : L \rightarrow \omega$ from the language into the set of the non-negative integers that we call the **arity map**.

We usually refer to the signature only by naming the set L , but we always assume that we are also fixing its partition into functions and relations as well as the arity map.

1.1.2 Remarks.

We need not specify the nature of the elements of L ; they could be anything: they are just symbols, i.e. they will stand for something else. All what matters about

elements of L is the value of the arity map, and their being (members of the set of symbols for) functions, respectively, relations. Typically we shall use the letter f for functions and the letter r for relations so we will not need to specify that $f \in L_{\text{function}}$ and $r \in L_{\text{relation}}$. When $f, r \in L$ we call $\text{Arity}(f)$, $\text{Arity}(r)$ the arity of the function f , respectively, the relation r .

1.1.3 First-order structures.

A **(first-order) structure** of signature L is a pair that consists of

- 1 a set M , called the **domain** (also the **support** or the **currier**); and
- 2 a mapping, called the **interpretation** of L , that assigns:
 - a to each symbol for relation $r \in L$ a relation $r^M \subseteq M^n$ of arity $n = \text{Arity}(r)$ which is called the **interpretation of r in M** ; and
 - b to each symbol for function $f \in L$ a total function $f^M : M^n \rightarrow M$ of arity $n = \text{Arity}(f)$ which is called the **interpretation of f in M** .

We shall name structures after their domains. Below structure will always mean first-order structure. The specification *first-order* is used in these notes only for emphasis (we shall never encounter second-order structures).

1.1.4 Models.

In the literature the word **model** is often used as a synonym of *structure*. But here we prefer to reserve this word for structures that belong to some class that will be fixed in the context (for instance, in most of the chapters by *model* we will understand *elementary substructures of the universe*).

1.1.5 Constants.

Some authors introduce in the language a third syntactic category: constants. They stipulate that the language L is the disjoint union of three sets L_{relation} , L_{function} , and L_{constant} . The interpretation maps symbols in L_{constant} into elements of the structure. Notationally it is more convenient to think of constants as degenerate case of functions. In this notes **constants** are simply function symbols of arity 0. By convention, M^0 is the set $\{\emptyset\}$ so the interpretation of a constant c is completely determined by $c^M(\emptyset)$: the image of \emptyset under the map c^M . This element is usually denoted simply by c^M or, when M is clear from the context, c . The interpretation of the symbols for functions is required to be a total function, so the domain of a structure need at least to contain the interpretation of the constants.

1.1.6 Tuples.

Lower case letters a, b, c , etc. from the beginning of the Latin alphabet are used to denote elements of structures. When overlined the same letters denote arbitrary (finite) **tuples** of elements, for instance, \bar{a} stands for $a_0 \cdots a_{n-1}$ where n is called **the arity of \bar{a}** . Since we work with tuples all the time, we lighten the notation by leaving the arity unspecified when this is less relevant. Formally, a tuple \bar{a} of elements of A is a function from n , i.e. the set $\{0, \dots, n-1\}$, to A . This n may be 0; in this case \bar{a} is the empty map \emptyset . Sets and sets of tuples are denoted with upper case letters A, B, C , etc. from the beginning of the Latin alphabet. When overlined the same letters denote the Cartesian power of that set, i.e. \bar{A} stands for A^n , where again if possible we let n unspecified. When A itself is a set of tuples, \bar{A} is the set containing concatenations $\bar{a}_1 \cdots \bar{a}_n$ of tuples in A . A few times we need also to consider infinite tuples. Infinite tuples are sequences $\langle a_\alpha : \alpha < \lambda \rangle$, where λ is an infinite ordinal (typically a cardinal) that we call the arity of the tuple. Moving from finite to infinite tuples is not always unproblematic in model theory. So we prefer to use a distinguished notation for infinite tuples: we use the symbol $\bar{\bar{a}}$ which is non-standard; the arity of the tuple is usually left implicit as in the finite case. Tuples are by default finite.

1.1.7 Notation and warnings.

The set of elements occurring in a tuple is called the **range of the tuple**; we denote it by $\text{rng } \bar{a}$. We abbreviate $A \cup \text{rng } \bar{a}$ with $A + \bar{a}$ and, in general, when the context suggests (e.g. as in $A \setminus \bar{a}$ or $\bar{a} \subseteq A$) one should understand $\text{rng } \bar{a}$ for \bar{a} . The symbols $A\bar{a}$ and $\bar{A}\bar{a}$ are used to mean the set of tuples obtained concatenating an element of A , respectively, a tuple of \bar{A} , with the tuple \bar{a} . We write AB for the set of tuples obtained concatenating the tuples in A with tuples in B . The same notation AB is also used to denote $A \cup B$, the context will disambiguate. Similar notations and conventions are used for infinite tuples.

1.1.8 Substructures.

Let M and N be two structures with the same signature L . We say that **M is a substructure of N** if

- 1 $M \subseteq N$, that is, the carrier of M is contained in the carrier of N ; and
- 2 the interpretation of relations and functions in M is the restriction to M of the interpretation in N , that is,
 - a for every relation of the language $r^M = r^N \cap M^n$, where n is the arity of r ; and
 - b for every function $f^M = f^N \upharpoonright M^n$, where n is the arity of f .

We write $M \preceq_{\text{qf}} N$ when M is a substructure of N . The meaning of the symbol will be clear in 3.1.3 below. This is only a temporary notation, soon we will set a context where we can unambiguously write $M \subseteq N$ for $M \preceq_{\text{qf}} N$.

1.1.9 Generated substructures.

Let M be a structure of signature L . If L is relational, given a subset $A \subseteq M$ we obtain a substructure with domain A by simply restricting the relations of M to A . But when L is not relational, an arbitrary subset of M need not be the domain of a substructure: it is required that $f^M[A] \subseteq A$ for every $f \in L$. Still it is easy to see that any subset $A \subseteq M$ uniquely determines a least substructure of M containing A : the **substructure of M generated by A** . This exists because the intersection of (the domains of) an arbitrary family of substructures of M is again (the domain of) a substructure of M . If there are no functions in the language i.e., when the language is **relational**, the (domain of the) structure generated by A is A itself, in general, it is the least subset of M closed under (the interpretation in M of) the functions of the language. That is, the set

$$* \quad \bigcup_{i \in \omega} A_i \quad \text{where} \quad A_0 = A \quad \text{and} \quad A_{i+1} = A_i \cup \left\{ f^M \bar{a} : \bar{a} \subseteq A_i, f \in L \right\}.$$

(In the notation above we assume that the arity of f and \bar{a} match.) It is clear that the cardinality of the structure generated by A is at most $|L A \aleph_0|$. We write $\langle A \rangle_M$ and $\langle \bar{a} \rangle_M$ for the substructure of M generated by A , respectively, $\text{rng } \bar{a}$. When M is clear from the context we omit it.

1.1.10 The prime substructure.

The substructure of M generated by the empty set is called the **prime substructure** of M . We say that two structures have the same **characteristic** if their prime substructures are isomorphic (isomorphisms are defined below). This terminology is non-standard; it is borrowed from field theory. With the notation above: when A_0 is the empty set then A_1 is the set $\{f^M \emptyset : f \in L\}$ (in (*) above we read $\bar{a} \subseteq \emptyset$ as saying that \bar{a} is the empty tuple). So A_1 contains just the constants of M . So the prime substructure of M is the substructure generated by the (interpretation of the) constants. When L contains no constants the substructure generated by the empty set is empty.

1.1.11 Maps.

Some concepts in model theory are easier to formalize using partial functions between structures (for instance, the notions of *partial isomorphism* and of *elementary map* in 3.2.1 below). Here partial functions are called for short: maps. Precisely, a **map** is a triplet that we denote by $F : M \rightarrow N$ where:

- 1 M is a structure that we call the **domain** of the map;
- 2 N is a structure that we call the **codomain** of the map;
- 3 F is a function from a subset of M , that we call the **domain of definition of F** and denote by **dom F** , onto a subset of N , that we call the **range of F** and denote by **rng F** .

A map $F : M \rightarrow N$ is **total** if the domain of definition coincides with the domain and it is **surjective** if the range coincides with the codomain. If $a \in \text{dom } F$ we write Fa for the image of a under F and, if $A \subseteq M$, we write FA for the image of A under F , that is, the set $\{Fa : a \in A \cap \text{dom } F\}$. (When FA is the domain of a substructure of N then FA may denote this structure.) When this notation is ambiguous we use delimiters: we write $F(a)$, respectively, $F[A]$.

The composition of two maps $F : M \rightarrow N$ and $H : N \rightarrow K$ is the map $HF : M \rightarrow K$, where HF is the composition of the functions H and F . The composition of two maps is only defined when the codomain of the first map is the domain of the second. When F is injective, we define the **inverse** of the map $F : M \rightarrow N$ to be the map $F^{-1} : N \rightarrow M$. We say that the map **$F' : M' \rightarrow N'$ extends $F : M \rightarrow N$** if M' and N' are superstructures of M , respectively, N , and the function F' an extension of F .

1.1.12 Isomorphisms, embeddings and automorphisms.

Let M and N be two structures with the same signature L . Let $F : M \rightarrow N$ be a total, injective map from the domain of M to the domain of N . We say that **F is an embedding of M in N** if

- 1 $f^N(F\bar{a}) = F(f^M\bar{a})$ for every function $f \in L$ and every tuple \bar{a} of elements of M ; and
- 2 $F\bar{a} \in r^N \Leftrightarrow \bar{a} \in r^M$, for every relation $r \in L$ and every tuple \bar{a} of elements of M .

Note that the symbol F is used to denote also the function that maps the tuple $a_1 \cdots a_n$, denoted by \bar{a} , to the tuple $Fa_1 \cdots Fa_n$. We are also assuming that the arity of the tuple \bar{a} matches the arity of f , respectively, r . Note that, when \bar{a} empty, the first clause says that each constant of M is mapped in the corresponding constant of N . We say that **M and N are isomorphic** if there is an embedding $F : M \rightarrow N$ which is surjective. Note that the inverse of an isomorphism is also an isomorphism. An isomorphism $F : M \rightarrow M$ is called an **automorphism of M** .

1.1.13 Parameters.

The definitions above **localize** to a set of **parameters**. (Parameters are simply elements of some structure, the use of the word is emphatic.) Let $F : M \rightarrow N$ be a map and let $a \in \text{dom } F$. If $Fa = a$ we say that **F fixes a** . Let $A \subseteq \text{dom } F$; we say

that **F fixes A pointwise** if F fixes every element of A . Let M and N be structures containing the set A . We say that $F : M \rightarrow N$ is an isomorphism **over A** if it is an isomorphism that fixes A pointwise. Embeddings, automorphisms, and partial isomorphisms **over A** are defined similarly. Sometimes we use the abbreviations **A -isomorphism**, **A -embeddings**, and **A -automorphisms**.

1.1.14 An exercise.

Prove that if $F : M \rightarrow K$ and $H : N \rightarrow K$ are embeddings then M and N have the same characteristic. In 11.3.5 we shall see that if M and N have the same characteristic then there are two embeddings as above. EXERCISE

1.1.15 Chains.

Chains of sets, functions, structures etc. are used frequently in these notes. We fix some terminology. Let λ denote an ordinal; a **chain** of sets of length λ is a map $\alpha \mapsto A_\alpha$ defined for every $\alpha < \lambda$ such that $A_\alpha \subseteq A_{\alpha+1}$ and, when α is a limit ordinal,

$$A_\alpha = \bigcup_{\beta < \alpha} A_\beta.$$

The **limit** of the chain is the union of all elements of the chain: this is denoted by **A_λ** . A chain of functions is identified with the chain of their graphs. A chain of structures is defined similarly replacing \subseteq with \preceq_{qf} ; as the union of a chain of structures is again a structure, the limit stages are well-defined. Note that for us chains of sets, functions, and structures are always **continuous**, that is, they are union at limit ordinals (sometimes we may restate this for emphasis).

1.2 Examples

Structures occur everywhere in mathematics but their signature may not always be presented explicitly. Sometimes there is more than one natural choice. However note that to speak about isomorphisms and/or substructures one needs to disambiguate.

1.2.1 The signature of a field.

To a field F one usually associate the structure $\langle F, +, -, \cdot, 0, 1 \rangle$. Though we did not introduce this notation formally, it should be clear which structures we have in mind: F is the carrier and $+$, $-$, \cdot , 0 , and 1 are the functions; there is no relation. This is actually called is called **the signature of rings**. The arities of the functions are: $\text{Arity}(+) = \text{Arity}(\cdot) = 2$, $\text{Arity}(-) = 1$ and $\text{Arity}(0) = \text{Arity}(1) = 0$. (Note parenthetically that this notation confuses, as we often do, the symbols in L with the typographic characters commonly used to denote the functions and

relation that interpret the symbols in L .) In a field it is also natural to have a function that gives the multiplicative inverse of non-zero element. (Strictly speaking this is not a function because it is undefined in 0 and we stipulated to interpret functions symbols with total functions. But we can agree that, e.g. $0^{-1} = 0$.) Note that the substructures of $\langle F, +, -, \cdot, ^{-1}, 0, 1 \rangle$ are fields while the substructures of $\langle F, +, -, \cdot, 0, 1 \rangle$ are just rings.

1.2.2 The signature of a group.

Also for groups there is more than one possible choice. The default signature is $\langle G, \cdot, ^{-1}, 1 \rangle$. When the group is commutative, this may also be denoted by $\langle G, +, -, 0 \rangle$. With respect to this signature the substructures are exactly the subgroups.

1.2.3 Graphs.

A **graph** (in its simplest setting) is a structure with a single binary (i.e. of arity 2) relation r which is symmetric and irreflexive (no element is in relation with itself). Elements of the domain are called **vertexes**; pairs in the relation are called **links**.

1.2.4 Dense linear orders.

A **linear order** is a structure M with in its signature only a binary relation $<$ which is:

- 1 transitive: if $a < b < c$ then $a < c$;
- 2 irreflexive: $a \not< a$ for every a ; and
- 3 antisymmetric: if $a \neq b$ then either $a < b$ or $b < a$.

An order is **dense** if

- 1 it contains at least two elements;
- 2 whenever $a < b$ there is an element c such that $a < c < b$.

Finally, a linear order has no **endpoints** if it has no first and no last element. The rational numbers Q with the natural order is a dense linear order without endpoints.

1.2.5 Vector spaces.

There are at least two natural ways to view a vector space as a structure. We present here the standard one: the field of scalars is formalized in the signature the domain contains only the additive group. A second possibility would be to view a vector space as a two sorted structure composed of a field, an additive group, and an action of the field on the group (but this is rarely used). The language L_K of vector spaces over K contains: a constant 0; a binary functions $+$ and $-$; and, for every element k of the field K , a unary function f_k . A vector space V over the field of scalars K

is a structure of signature L_K . The interpretation of 0, $+$, and $-$ is obvious while f_k is interpreted in the function that maps the vector v to kv .

Chapter 2

First-order formulas

We associate to every structure an algebra of sets (of tuples). This is obtained by taking intersection, complementation and projections of the relations given in the language. If the language has functions we also require the algebra to be closed under inverse image. But this is not all. We want a canonical way for naming these sets. The naming we use is the most natural: it is given by the order in which relations, functions, and the operations of intersection, complementation, and projections have been composed. These names are what we call first-order formulas.

Notation and global assumptions.

Fix a signature L and a structure M of signature L . We use the symbols A, B , etc. to denote sets of parameters.

2.1 The syntax

Terms are the *names* that we attach to the composition of the functions of L . All what the names need to do is record the order in which function have been composed. We begin defining terms then we move to the definition of formulas (which follows the same line). But first of all we define what a *name*, i.e. a word in some fixed alphabet, is.

2.1.1 Variables.

Fix an infinite set V that we call the set of **(free) variables**; think of these as placeholders for inputs of functions and relations. In most arguments countably many free variables suffices but in a few cases we need much more. So we assume that V is always as large as necessary. We use the letters x, y, z, \dots to denote elements of V .

2.1.2 Parameters.

Fix an arbitrary set A that we call the set of **parameters**. Parameters are elements of some structure(s) that will occur in terms and formulas.

2.1.3 The definition of term.

Formally **terms** are words on the alphabet containing LVA plus two auxiliary elements, that we denote by the symbols \sqcup and \sqcap . These last two objects have the role of delimiters (we could have used normal parenthesis instead but, at least for this chapter, we want to stress the difference between a symbolic expression and its interpretation). Terms are defined by induction as follows:

- 1 each free variable and each parameter is a term; and
- 2 if \bar{t} is a tuple of terms and f is a function symbol with the same arity as \bar{t} then $f\sqcup\bar{t}\sqcap$ is a term. By $f\sqcup\bar{t}\sqcap$ we mean the word obtained by concatenating $t_1 \dots t_n$, prefixing it with the two symbols $f\sqcup$, and finally postfixing it with \sqcap .

Terms obtained from 1 above, that is, terms that are either free variables or parameters are called **atomic terms** (sometimes in the literature also constants are called atomic terms). Terms where no free variable occurs are called **closed terms**. When we want to stress that all parameters occurring in a term t are in A then we may say that t is a **term over A** or **A -term**; \emptyset -terms are also called **parameter-free terms**.

2.1.4 Substitutions of terms for variables.

When \bar{s} is a tuple of terms with the same arity as \bar{x} we write $t(\bar{x}/\bar{s})$ for the term obtained substituting \bar{s} for \bar{x} in t coordinatewise. That $t(\bar{x}/\bar{s})$ is a term is a claim which, strictly speaking, needs to be proved: this we do in the proposition below. It is convenient to display in the notation the free variables occurring in a term. So we write $t(\bar{x})$ to mean that *all* free variables of t are among \bar{x} (note that \bar{x} may contain more variables). To denote the substitution of \bar{s} for \bar{x} in $t(\bar{x})$ (i.e. when all variables of t have been declared) we often use the abbreviated notation $t(\bar{s})$.

PROPOSITION Let $t(\bar{x})$ be a term (so, with free variables among \bar{x}). Let \bar{s} be a tuple of terms. Then $t(\bar{s})$ (that is, the result of substituting coordinatewise \bar{s} for \bar{x} in t) is a term.

PROOF The claim is proved by induction on the syntax of t . That is, checking that the claim is true for t is atomic (which is immediate) and proving that if the claim holds for all the terms in the array \bar{t} then it holds for the term $f\sqcup\bar{t}\sqcap$ (which is also immediate).

2.1.5 Logical connectives.

Just as terms name functions, formulas name sets of tuples. We introduce simultaneously also a restricted class of formulas, the quantifier-free formulas. The **quantifier-free formulas** are words on the alphabet containing LVA together with the symbols: \perp , \sqcup , $=$, \perp , \neg , and \wedge . The last three are **logical connectives** these are called respectively **falsum**, **negation** and **conjunction**. The idea is to formalize the concept of empty set, intersection, and complementation. If we read \neg and \wedge with *is not* and *and* the intended meaning of the expressions defined below should be clear. To introduce quantifiers we need to use a countable set U of auxiliary variables disjoint of V . The reason is technical and is discussed in 2.1.8 below. Variables in U are called **bound variables**. A **formula** (non-necessarily quantifier-free) is a word on the alphabet containing $LUV A$, the symbols above, and the symbol \exists which is called **existential quantifier**. Here is the inductive definition of formula.

- 1a If \bar{t} is a tuple of terms and r is a relation symbol with the same arity as \bar{t} , then $r\bar{t}$ is a formula;
- 1b if t and s are terms then $t = s$ is an formula;
- 1c \perp is a formula;
- 2a if φ and ψ are formulas, then $\neg\varphi$, $\varphi \wedge \psi$ are formulas;
- 2b if φ is a formula, x is a free variable, and u is a bound variable not occurring in φ , then $\exists u \varphi(x/u)$ is a formula (where $\varphi(x/u)$ denotes the literal substitution of u for x in φ).

Formulas of the form 1a or 1b are called **atomic**. Formulas obtained without the use of 2b are called **quantifier-free formulas**. When we want to specify that the parameters the formula come from some set A , then we say **formula over A** or for short **A -formula**. In particular, \emptyset -formulas are parameter-free. A formula without free variables (i.e. all its variables occur under the scope of a quantifier) is called a **closed formula** or a **sentence**.

2.1.6 Bound versus free variables.

Let us declare right away that from the next section we shall use a more tolerant notation: we shall make no distinction between bound and free variables and simply write $\exists x \varphi$ for $\exists u \varphi(x/u)$. In fact, working all the time with two distinct sorts of variables would overload the notation. We introduced the set of variables U for a technical reason: if we had not kept bound and free variables distinct we would have been forced to give a less straightforward definition of substitution. Consider, for instance, the substitution of the parameters ab for xy in the formula $s \sqcup y \wedge \exists y r \sqcup x y \sqcup$. This does not yield any sensible formula! The result we would like to obtain instead is: $s \sqcup b \wedge \exists y r \sqcup a y \sqcup$, that is, we want to substitute a for y in the first conjunct while in the second conjunct we want to leave y unchanged since it occurs under the scope of the quantifier $\exists y$. Keeping bound variables distinct from free variables we get

around the problem: literal substitution of any term for free variables always yields a well-formed formula (cf. the proposition below).

2.1.7 First- versus second-order.

For us formula means first-order formula. The specification **first-order** is used for emphasis. We will never encounter *second-order* formulas: these would be formulas with extra sorts of variables that, in the intuition, are meant to range over subsets \bar{M} .

2.1.8 Substitutions of terms for free variables.

When \bar{x} is an array of free variables, \bar{t} an array of terms, and φ a formula, we write $\varphi(\bar{x}/\bar{t})$ for the literal and coordinatewise substitution of \bar{t} for \bar{x} in φ . As for terms, we usually introduce a formula together with a tuple of free variables and write: $\varphi(\bar{x})$. We agree that when we use this expression we are displaying *all* free variables of φ . To denote the substitution of \bar{s} for \bar{x} in $\varphi(\bar{x})$ we use the abbreviated notation $\varphi(\bar{s})$. The following technical proposition proves that substitutions of terms for variables in formulas works well. This we use implicitly in the definition of interpretation of a formula (cf. item 2b of Paragraph 2.2.8 with a parameter b for \bar{t}).

PROPOSITION Let $\varphi(\bar{x})$ be a formula (so, with free variables among \bar{x}). Let \bar{t} be a tuple of terms. Then $\varphi(\bar{t})$ (that is, the result of substituting coordinatewise \bar{t} for \bar{x} in φ) is a formula.

PROOF The claim is proved by induction on the syntax of φ . The claim for atomic formulas follows directly from the analogous claim for terms 2.1.4 above. Induction for the connectives \neg and \wedge is straightforward. So it suffices to prove that if the claim holds for φ it holds also for $\exists u \varphi(y/u)$, where y is a free variable and u is a bound variables not occurring in φ . Ideally, to prove that $\exists u \varphi(y/u)(\bar{x}/\bar{s})$ is a formula we would like to show that it equals $\exists u \varphi(\bar{x}/\bar{s})(y/u)$ and apply the induction hypothesis. But unfortunately, when y occurs in \bar{s} the two substitutions may not commute. Still, the reader can easily check that, in general, $\varphi(\bar{x}/\bar{t})(\bar{y}/\bar{s})$ equals $\varphi(\bar{y}/\bar{s})(\bar{x}/\bar{t})$ whenever the tuples \bar{x} and \bar{y} have no variable in common, no variable of \bar{x} occurs in \bar{s} and no variable of \bar{y} occurs in \bar{t} . So we only need an intermediate substitution: let w be a free variable not occurring in φ , \bar{t} nor \bar{x} . By induction hypothesis, $\varphi(y/w)$ is a formula: let denote it by φ' . Obviously, $\exists u \varphi(y/u)$ is literally the same formula as $\exists u \varphi'(w/u)$. Since w does not occur in \bar{x} nor in \bar{t} we obtain that $\varphi'(w/u)(\bar{x}/\bar{t})$ equals $\varphi'(\bar{x}/\bar{t})(w/u)$. By induction hypothesis, $\varphi'(\bar{x}/\bar{t})$ is a formula, so, by Definition 2.1.5 above, $\exists u \varphi'(\bar{x}/\bar{t})(w/u)$ is a formula. This proves the claim.

2.1.9 Infinitely many variables.

Formulas and terms are finite objects so at most finitely many free variables occur in a formula. Still nothing prevent us to write $\varphi(\bar{x})$ and $t(\bar{x})$ with the obvious meaning. This notation is convenient to deal with infinitely many formulas: we can use a unique infinite tuple of variables for all formulas.

2.1.10 An exercise.

We will often use the following fact. For every A -formula $\varphi(\bar{x})$ there is a parameter-free formula $\psi(\bar{z}\bar{x})$ and a tuple $\bar{a} \subseteq A$ such that $\varphi(\bar{x})$ is (literally) equal to $\psi(\bar{a}\bar{x})$. This can be easily proved by induction on the syntax of $\varphi(\bar{x})$. EXERCISE

2.2 The semantic

Finally we come to the interpretations of terms and formulas.

2.2.1 The interpretation of terms.

Let t be a closed term with parameters in M . We define by induction the **interpretation of t in M** which we denote by t^M :

- 1 if t is an atomic term (since t is closed it must be a parameter) then t^M is t itself; and
- 2 if t has been obtained as in 2 of Paragraph 2.1.3 above from the tuple of terms \bar{s} and the function symbol f , then t^M is the element $f^M(\bar{s}^M)$.

The definition above applies only to closed terms. In general, the interpretation of the term $t(\bar{x})$ is the function $t^M(\bar{x})$ that maps the tuple \bar{a} to $t^M(\bar{a})$.

2.2.2 Unique readability of terms.

Strictly speaking we should now prove that every term has a unique interpretation. The interpretation of a term could be non-well-defined if the array \bar{s} in (2) above was not uniquely determined by the term t . So we need to prove that if \bar{t} and \bar{s} are two sequences obtained concatenating the terms $t_1 \cdots t_n$, respectively, $s_1 \cdots s_n$ and if $\bar{t} = \bar{s}$ (as sequences) then $t_1 = s_1 \cdots t_n = s_n$. The delimiters \lrcorner and \llcorner now prove useful. In each term right parenthesis have to match left parenthesis: this gives a unique way to split the \bar{t} and \bar{s} into their components. We can safely skip the proof of this fact. (We could *not* if we had to write a computer program which recognizes terms!).

2.2.3 An example: polynomials.

Polynomials are the paradigmatic example of terms. In the field of real numbers $\langle R, +, -, \cdot, 0, 1 \rangle$ terms over \emptyset are (once interpreted) polynomial functions with integer coefficients and arbitrary many variables. The same is true for terms over the integers $Z \subseteq R$. Observe the difference between syntax and semantic: this second example is a strictly larger class of syntactic objects but, once interpreted in the structure R , it yields the same set of functions. The interpretation of terms over the rationals $Q \subseteq R$ yields polynomial functions with rational coefficients: a strictly larger class of functions.

2.2.4 A remark on the generated substructure.

Let $A \subseteq M$. The reader can prove as an exercise that the (domain of the) substructure generated by A in M contains exactly the elements of the form $t^M(\bar{a})$ for $\bar{a} \subseteq A$ and t a parameter-free term. (This requires a proof by induction on the syntax of t and on the construction of the substructure generated by A as described in 1.1.9 above.) EXERCISE

2.2.5 The role of equality.

Below we define the interpretation of formulas. Note the special status that equality has. Syntactically (cf. case 1b of Definition 2.2.6 below), equality behaves just like any binary relation symbol but its interpretation (cf. case 1b of Definition 2.2.8 below) is fixed: it does not depend on M . For this reason equality is usually considered to be a logical symbol and not a symbol of the language.

2.2.6 The interpretation of quantifier-free formulas.

Now we assign a **truth value** to closed quantifier-free formulas. As for terms the interpretation of a non-closed formulas is derived from that of closed formulas. Non-closed formulas are interpreted as sets as explained in the paragraph below. So, let φ be an quantifier-free formula without free variables and with parameters in M . We define when φ is **true (in M)**. If φ is not true we say that it is **false**. We also say that **M models φ** , respectively, does not model φ . We write $M \models \varphi$ when M models φ and $M \not\models \varphi$ when it does not.

- 1a If φ is the atomic formula $r\bar{t}\bar{t}$ then we stipulate that φ is *true* if and only if the tuple \bar{t}^M belongs to the relation r^M .
- 1b If φ is the atomic formula $\lrcorner t = s \llcorner$ then we stipulate that φ is *true* if and only if $t^M = s^M$.
- 1c The formula \perp is *false*.
- 2a If φ is of the form $\lrcorner \psi \wedge \xi \llcorner$ then we stipulate that φ is *true* if and only if both ψ and ξ are *true*. If φ is of the form $\neg\psi$ then we stipulate that φ is *true* if and

only if ψ is not *true*.

2.2.7 What does truth of quantifier-free sentences depends on ?

In the sequel we will frequently evaluate the same sentence in different structures. This is clearly possible: as soon as the structure N contains the parameters of φ we may view φ as an N -formula. Inspection of the definition above shows that the truth of a quantifier-free sentence only depends on the structure generated by its parameters. This is true also when φ is parameter-free: then the truth of φ depends on the characteristic (i.e. the prime substructure) of the structure in which it is evaluated. For instance, in the language of rings the quantifier-free sentence $1 + 1 + 1 = 0$ is true in rings of characteristic 2, false otherwise.

2.2.8 The interpretation of quantifiers.

As above we define only the interpretation of closed formulas; so let φ be an formula without free variables and parameters in M . We extend the definitions above with the following clause:

2b if φ has the form $\exists u \psi(y/u)$, then M models φ if and only if M models $\psi(y/b)$ for some b in M .

Note that in contrast to the quantifier-free case, to define the truth value of non-quantifier-free formulas we refer to the whole structure M , in fact, $\exists u$ will be interpreted as *there is an u in M* . Consequently the truth value of a non-quantifier-free formula is in general dependent on M .

2.2.9 Non-closed formulas.

The truth of formulas that are not closed is undefined: we can evaluate the truth of a formula $\varphi(\bar{x})$ only after we replace \bar{x} with a tuple \bar{a} from M with the same arity of \bar{x} . For easy of speech we will say that $\varphi(\bar{x})$ is **holds in M** if $M \models \varphi(\bar{a})$ for every tuple \bar{a} . If $M \models \varphi(\bar{a})$ for some tuple \bar{a} then we say that $\varphi(\bar{x})$ is **consistent in M** .

2.2.10 Abbreviations and easy going notation.

Trusting that, by now, the reader has clear the distinction between the syntax of terms and formulas and their interpretation, we adopt a more informal notation. For instance we use normal parentheses for \perp and \lrcorner ; also we omit the outermost parentheses. There are a number of other logical connectives that are introduced as abbreviations to keep formulas readable. E.g. we write $\varphi \vee \psi$ (read: φ or ψ) for $\neg(\neg\varphi \wedge \neg\psi)$. The connective \vee is called a **disjunction**. We do not use parenthesis around long conjunctions like $\varphi_1 \wedge \dots \wedge \varphi_n$ since these connective has an associative meaning; similarly for disjunctions. We write $\exists \bar{x} \varphi$ for $\exists x_1 \dots \exists x_n \varphi$, and $\bar{x} = \bar{y}$

for $x_1 = y_1 \wedge \dots \wedge x_k = y_k$. We abbreviate $\neg \exists \bar{x} \neg \varphi$ with $\forall \bar{x} \varphi$ which reads: φ holds for every \bar{x} . The logical connective \forall is called **universal quantifier**. We abbreviate $\neg \varphi \vee \psi$ with $\varphi \rightarrow \psi$ which reads: φ implies ψ . The connective \rightarrow is called **implication**. Semantically implication corresponds to inclusion: the formula $\forall \bar{x} [\varphi(\bar{x}) \rightarrow \psi(\bar{x})]$ holds in M if and only if $\varphi(\bar{M}) \subseteq \psi(\bar{M})$. We abbreviate $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ with $\varphi \leftrightarrow \psi$ which reads: φ is logically equivalent to ψ . Semantically this corresponds to equality: the formula $\forall \bar{x} [\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})]$ is true in M if and only if $\varphi(\bar{M}) = \psi(\bar{M})$. Finally, for every positive integer n we write $\exists^{\geq n} \bar{x} \varphi$ for

$$\exists \bar{u}_1 \dots \bar{u}_n \bigwedge_{1 \leq i < j \leq n} \bar{u}_i \neq \bar{u}_j \wedge \bigwedge_{1 \leq i \leq n} \varphi(\bar{x}/\bar{u}_i).$$

This says that there are more than n distinct tuples satisfying φ . We write $\exists^{=n} \bar{x} \varphi$ to abbreviate the formula $\exists^{\geq n} \bar{x} \varphi \wedge \neg \exists^{\geq n+1} \bar{x} \varphi$. Both the formal and the intuitive interpretation of these connectives should be clear.

2.3 Definable sets.

Finally we come to the semantic counter-part of the syntactic notions developed above: definable sets.

2.3.1 Definable sets.

Let $\varphi(\bar{x})$ be a formula. A tuple $\bar{a} \subseteq M$ such that $M \models \varphi(\bar{a})$ is called a **solution of $\varphi(\bar{x})$** or a **witness of $\exists \bar{x} \varphi(\bar{x})$** . We also add **in M** if it necessary to make clear where truth is evaluated. Let $\varphi(\bar{x})$ be a formula; we define

$$\varphi(\bar{M}) := \{ \bar{a} \subseteq M : M \models \varphi(\bar{a}) \}.$$

Sets of the form $\varphi(\bar{M})$ where $\varphi(\bar{x})$ is an A -formula are called **A -definable subsets of M** . We say simply **definable** for M -definable.

2.3.2 Type-definable sets.

Let $p(\bar{x})$ be an infinite set of formulas with free variables among \bar{x} , where this is a possibly infinite tuple of variables. When speaking about infinite sets of formulas, we often use the word *realization* for *solution*: we say that \bar{a} is a **realization of $p(\bar{x})$** when $M \models \varphi(\bar{a})$ for every formula $\varphi(\bar{x})$ in $p(\bar{x})$. We may also say that \bar{a} **realizes $p(\bar{x})$** ; when M is not clear from the context we may specify **in M** . We write $M \models p(\bar{a})$ for short. We say that **$p(\bar{x})$ is consistent in M** if it has a realization in M . We write

$$p(\bar{M}) := \{ \bar{a} \subseteq M : M \models p(\bar{a}) \}.$$

for the set of realizations of $p(\bar{x})$. In other words,

$$p(\bar{M}) = \bigcap_{\varphi \in p} \varphi(\bar{M}).$$

A set of formulas $p(\bar{x})$ is usually called a **type** sets of the form $p(\bar{M})$ are said to be **type-definable**.

2.3.3 Isomorphisms preserve definable sets.

The following easy fact says that definable sets live in the category of structures and isomorphisms between structures.

PROPOSITION Let M and N be two structures. Let $F : M \rightarrow N$ be an isomorphism. Then for every parameter-free formula $F[\varphi(\bar{M})] = \varphi(\bar{N})$.

PROOF Recall that

$$F[\varphi(\bar{M})] = \{F\bar{a} : \bar{a} \subseteq M \text{ and } M \models \varphi(\bar{a})\}$$

and

$$\varphi(\bar{N}) = \{\bar{b} \subseteq N : N \models \varphi(\bar{b})\}.$$

Since $F : M \rightarrow N$ is surjective, we have

$$\varphi(\bar{N}) = \{F\bar{a} : \bar{a} \subseteq M \text{ and } N \models \varphi(F\bar{a})\}.$$

So it is sufficient to prove that

$$* \quad M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(F\bar{a})$$

for every $\bar{a} \in M$. Actually it suffices to prove one direction (say, left to right) of the implication in (*) because the converse follows by substituting $\neg\varphi(\bar{x})$ for $\varphi(\bar{x})$.

We proceed by induction on the syntax of the formula. Preliminary to that, we need to show that for every term t and for every \bar{a}, c in M if $t^M(\bar{a}) = c$ then $t^N(F\bar{a}) = Fc$. This is again proved by induction of the syntax; from it it follows easily that (*) holds for every atomic formula $\varphi(\bar{x})$. The proof of these facts and the proof of the induction step for the Boolean connectives (which is straightforward) is entrusted to the reader. So we only prove the induction step for existential quantifiers, that is, the equivalence in (*) holds for $\exists y \varphi(y\bar{x})$ assuming it holds for the formula $\varphi(y\bar{x})$. Let $\bar{a} \subseteq M$ be such that $\exists y \varphi(y\bar{a})$ is true in M . Then there is a b such that $\varphi(b\bar{a})$ holds in M . By induction hypothesis $\varphi(FbF\bar{a})$ is true in N . Then $\exists y \varphi(yF\bar{a})$ is consistent in N as required.

2.3.4 A degenerate case.

When \bar{x} is an empty tuple, $\varphi(\bar{M})$ is either \emptyset or $\{\emptyset\}$ according to whether φ is true or false in M . (For those who find this notation *too* degenerate: take \bar{x} of positive arity but let φ be a sentence, then $\varphi(\bar{M})$ is either \emptyset or \bar{M}) So Proposition 2.3.3 above says that if M and N are isomorphic then $M \models \varphi$ if and only if $N \models \varphi$.

2.3.5 Isomorphisms preserve type-definable sets.

It is immediate that Proposition 2.3.3 above holds also if we substitute $p(\bar{M})$ and $p(\bar{N})$ for the sets $\varphi(\bar{M})$, respectively, $\varphi(\bar{N})$.

2.3.6 Mapping a definable set = mapping its definition.

The following generalization of the theorem above will be frequently used in the chapters that follow. The proof is left to the reader as exercise. Let $F : M \rightarrow N$ be an isomorphism and fix a tuple $\bar{a} \subseteq M$. Let $\varphi(\bar{x}\bar{z})$ be a parameter-free formula. Write $\psi(\bar{M}\bar{a})$ for the subset of \bar{M} defined by $\varphi(\bar{x}\bar{a})$ and $\psi(\bar{M}F\bar{a})$ for the subset of \bar{M} defined by $\varphi(\bar{x}F\bar{a})$. Then $F[\psi(\bar{M}\bar{a})] = \psi(\bar{N}F\bar{a})$. In particular, if A is a common subset of M and N and $F : M \rightarrow N$ is an A -isomorphism then for every A -formula $F[\varphi(\bar{M})] = \varphi(\bar{N})$. EXERCISE

2.3.7 A trivial example: pure equality.

When the language L is empty, formulas are formed using only the symbol of equality. Every set is a structure and every partial injection is a partial isomorphism. Two structures are isomorphic if and only if they have the same cardinality. Let M be a structure with empty signature. The definable subsets of M are easy to describe. For simplicity we restrict the attention to definable sets of arity one: these are exactly all finite and cofinite (i.e. with a finite complement) subsets of M . One direction of the inclusion is immediate: let C be the finite set $\{c_1 \dots c_n\}$; the formula $x = c_1 \vee \dots \vee x = c_n$ defines C ; its negation defines $M \setminus C$. To prove the converse let $\varphi(x)$ be a formula and suppose that all the parameters occurring in $\varphi(x)$ are in C . We claim that $\varphi(M) \subseteq C$ or $\neg\varphi(M) \subseteq C$. Suppose not for a contradiction: there are $a, b \in M \setminus C$ such that $\varphi(a)$ and $\neg\varphi(b)$. The map F that fixes all points in $M \setminus \{a, b\}$ and switches a with b is a bijection of M to M which fixes $c_1 \dots c_n$. So it is a C -automorphism. This contradicts 2.3.3 above.

Chapter 3

Elementarity

Finally, we begin using formulas and set of formulas to compare structures.

Notation and global assumptions.

Fix a signature L . Below **structure** stands for structure of signature L . We use the symbols M, N, K , ect. for structures and A, B, C , etc. for sets of parameters (i.e. elements of some structure).

3.1 Theories and elementarity

Two structures with the same signature may have very little in common. Two isomorphic structures are *practically* the same. Between this very loose and very strong degree of similarity there is another very important sort of equivalence: being indistinguishable by a first-order sentence. This is the relation of elementary equivalence.

3.1.1 Theories.

A set of sentences is **consistent** if there is a structure M such that $M \models T$, that is, $M \models \varphi$ for every $\varphi \in T$. A consistent set of A -sentences set T is called a **theory over A** or simply a *theory* when A is empty. We say that a theory is **complete for A -sentences** if for every A -formula φ either $\varphi \in T$ or $\neg\varphi \in T$. When the set A is not mentioned we understand it is the set of parameters that occurs in T .

Let M and N be structures and let $A \subseteq M, N$. The set of A -sentences that hold in M is called the **theory of M over A** and is denoted by $\text{Th}_A M$. The set of quantifier-free A -sentence hold in M is called the **quantifier-free theory of M over A** and is denoted by ${}_{\text{qf}}\text{Th}_A M$. When A is empty we omit it from the notation and the terminology. Note that $\text{Th}_A M$ is **complete for A -sentences**, that is,

for every A -sentence φ either $\varphi \in \text{Th}_A M$ or $\neg\varphi \in \text{Th}_A M$. Observe that, since the theory of a structure cannot contain both a formula and its negation, the two inclusions $\text{Th}_A M \subseteq \text{Th}_A N$ and $\text{Th}_A N \subseteq \text{Th}_A M$ are equivalent.

3.1.2 The characteristic of a structure.

The reader can prove as an exercise that M and N have the same characteristic (this has been defined in 1.1.10 above) if and only if ${}_{\text{qf}}\text{Th} M = {}_{\text{qf}}\text{Th} N$. So sometimes we call ${}_{\text{qf}}\text{Th} M$ the characteristic of M . EXERCISE

3.1.3 The relation of elementarity.

Let M and N be structures and let $A \subseteq M, N$. We say that **M and N are elementary equivalent over A** if M and N have the same A -theory, that is, M and N model the same A -sentences. We write $M \equiv_A N$. We write $M \equiv_{\text{qf}, A} N$ if M and N have the same quantifier-free theory over A . Again when A is the empty set we omit it from the notation and the terminology. Observe that two structures isomorphic over A are elementary equivalent over A . This follows from 2.3.3 above: just let the tuple of variables in $\varphi(\bar{x})$ be empty.

We write $M \preceq N$ when $M \subseteq N$ and $M \equiv_M N$. This is the same as requiring that $N \models \text{Th}_M M$ or that $M \models \text{Th}_M N$. In words we say that **M is an elementary substructure of N** or that **N is an elementary superstructure of M** . It is clear that the relation of being elementary substructure is transitive. The quantifier-free version of the notion of elementary substructure is simply the notion of substructure, in fact, $M \preceq_{\text{qf}} N$ (as defined in 1.1.8 above) just tells that N and M model the same atomic M -formulas. So it easily follows that they model the same quantifier-free M -formulas, that is, $M \equiv_{\text{qf}, M} N$.

3.1.4 Definable sets and elementarity

Let $M \preceq_{\text{qf}} N$. The reader can prove as an exercise that the following are equivalent:

- 1 $M \preceq N$,
- 2 $\varphi(\bar{M}) = \varphi(\bar{N}) \cap \bar{M}$ for every parameter-free formula $\varphi(\bar{x})$,
- 3 $\varphi(\bar{M}) = \varphi(\bar{N}) \cap \bar{M}$ for every M -formula $\varphi(\bar{x})$. EXERCISE

3.1.5 Recommendations

The terminology may suggest the following identity: substructure + elementary equivalent = elementary substructure. But this is far from being true. For an easy counter-example let L contain just the binary relation $<$, let M and N be the natural numbers, respectively the positive natural numbers. We interpret $<$ in M and N

with the natural order. Clearly $M \preceq_{\text{cf}} N$. It is also obvious that M and N are isomorphic structures so $M \equiv N$. But we do not have $M \preceq N$, in fact N models $\exists x(x < 1)$ while M does not.

3.1.6 Moral support

Some readers may feel that a concrete intuition is missing of what the theory of a structure is. It is also difficult to grasp what it means that a structure is an elementary substructure of another. The concern is justified: in most cases it is almost impossible to have a good grip of what the elements of $\text{Th}M$ are. For instance, if M is structure of the natural numbers with $+$ and \cdot as functions, then all theorems of number theory (both the known as the yet unknown) are contained in $\text{Th}M$! So to work with the relation elementarity (or the relation of elementary substructure) we often need to use an indirect approach. For the time being the only way we have to prove that two structures are elementary equivalent is to prove that these are isomorphic. Other techniques will be introduced in the next chapters.

3.1.7 Examples and non-examples

We give an example of elementary substructure. Let L contain just the binary relation $<$. We write N for the rational numbers with their natural order. Let M be the substructure of N with domain the open interval $(0, 1)$. Clearly M and N are isomorphic structures. We claim that $M \preceq N$. We need to show that $M \equiv_M N$ but it is clearly sufficient to show that $M \equiv_A N$ for every finite set $A \subseteq M$. This second claim is easily seen to be true: M and N are isomorphic over any finite set $A \subseteq M$ so elementary equivalence follows.

Now we give a non-example, let N be the rational closed interval $[0, 3]$ and let M be the interval $[1, 2]$ (with the same language as above). Again, there is an obvious isomorphism between M and N so, in particular, they are elementary equivalent. We claim that $M \not\preceq N$. Clearly, there is a parameter-free formula $\varphi(x)$ such that $M \models \varphi(1)$ while $N \not\models \varphi(1)$: the formula that says that 1 is the least element of the domain. So $M \not\preceq N$.

3.1.8 The elementary chain lemma.

Assume the notation about chains that has been introduced in 1.1.15 above. A chain of structures is called an **elementary chain** if $M_\alpha \preceq M_\beta$ for every $\alpha < \beta < \lambda$. The reader can easily check that for a chain to be elementary it suffices that $M_\alpha \preceq M_{\alpha+1}$ whenever $\alpha + 1 < \lambda$. This is proved by transfinite induction using the transitivity of the elementarity relation and the following proposition.

PROPOSITION Let M_α be a elementary chain of length λ , where λ is a limit ordinal. Let

M_λ be the limit of the chain. Then $M_\alpha \preceq M_\lambda$ for every $\alpha < \lambda$.

PROOF By induction on the complexity of the formulas. We only verify induction for the existential quantifier and leave the rest to the reader. So let $\varphi(x)$ be a M_α -formula. Suppose that $M_\alpha \models \exists x \varphi(x)$. Then $M_\alpha \models \varphi(a)$ for some $a \in M_\alpha$. By induction hypothesis $\varphi(a)$ holds in the limit of the chain hence $\exists x \varphi(x)$. Vice versa, assume that $\exists x \varphi(x)$ holds in M_λ , so for some c we have $M_\lambda \models \varphi(c)$. But $c \in M_\beta$ for some $\beta < \lambda$ so, by induction hypothesis, we have that the $M_\beta \models \varphi(c)$. So $M_\beta \models \exists x \varphi(x)$ and, by elementarity, $M_\alpha \models \exists x \varphi(x)$.

3.1.9 Finite structures: elementary \rightarrow isomorphic.

As remarked above, two isomorphic structures are elementarily equivalent. The converse implication is true for finite structures. This is actually bad news, it tells us that elementarity is too strong to produce an interesting relation of similarity between finite structures. However limiting in some way the strength of first-order formulas one can still recover a non-trivial relation. For instance one can require that the size of the sentences used to compare two finite structures has to be small in comparison with the size of the structures. In this way we quickly leave the realm of classical model theory and enter a field that is near in spirit and flavor to finite combinatorics and probability. For this reason we shall limit ourself (each time the distinction matters) to infinite structures. The proof that two elementary equivalent finite structures are isomorphic is postponed to 3.2.12 below.

3.1.10 Elementary equivalence versus isomorphism: categoricity.

For infinite structures the situation is radically different: elementary equivalence give a much coarser classification of structures than isomorphism. We will see that every infinite structures has elementary equivalent superstructures of arbitrary large cardinality (upward Löwenheim-Skolem theorem). Structures with different cardinality clearly cannot be isomorphic.

Besides that, we will see that there exist also non-isomorphic but elementarily equivalent structures of the same cardinality. Actually, for every infinite cardinal λ we can find 2^λ elementary equivalent structures of cardinality λ that are pairwise non-isomorphic.

A complete theory that has, up to isomorphism, only one structure of cardinality λ is called **λ -categorical**. We may say **countably categorical** for ω -categorical. Theories that are λ -categorical for every infinite cardinal λ are called **totally categorical**.

We will see that (for countable theories) it suffices that a theory is categorical in some uncountable cardinal to be categorical in all uncountable cardinals. That

is, there are only four possibilities for a countable theory: countably categorical, uncountably categorical, totally categorical, or categorical for no cardinality at all. This is the celebrated Morley theorem.

3.1.11 Countable dense linear orders.

We give an example of ω -categorical theory: the dense linear orders without endpoints. The proof is Cantor's; it is the paradigmatic example of back-and-forth construction a technique that we shall encounter often in the sequel.

PROPOSITION Let M and N countable dense linear orders without endpoints. Then M and N are isomorphic.

PROOF Fix some enumerations $\{a_i\}_{i \in \omega}$ and $\{b_i\}_{i \in \omega}$ of M , respectively, N . The isomorphism $F : M \rightarrow N$ is the union of the partial maps F_i which we now define by stages (to be precise, we define the graph of these maps). We begin with setting F_0 to be the empty map. At stage $i + 1$ we apply one of two procedures depending on the parity of i . When i is even we consider the least $h \in \omega$ such that a_h is not yet in the domain of F_i . Let A be the set $\{a \in \text{dom } F_i : a < a_h\}$. By the density of the order of N and the absence of the first and last elements, there is some b_k which is larger than all elements in $F_i[A]$ and smaller than all elements in $\text{rng } F_i \setminus F_i[A]$. Let F_{i+1} be the map $F_i + \langle a_h, b_k \rangle$. Observe that if F_i was order-preserving so is F_{i+1} . For i odd the construction is symmetric: pick an element b_k of N not yet in the range of F_i and pair it with some a_h in the domain M so that F_{i+1} is order-preserving. Let F be the union of F_i for $i \in \omega$. By induction on i it is easy to prove that every a_i is in the domain if F and every b_i is in the range of F . So F is total and surjective; F is order-preserving because it is the union of order-preserving maps.

3.1.12 Uncountable dense linear orders.

We give an example of 2^λ elementary equivalent but non-isomorphic structures of cardinality λ . We sketch the construction of these structures but we will not prove that these are elementary equivalent (this requires elimination of quantifiers, see 5.3.3 below).

The language contains only the symbol $<$; all structures will be dense linear orders without endpoints. As λ we can take an arbitrary uncountable cardinal. The proof is based on the following simple idea: a copy of the real line followed by a copy of the rational line is not isomorphic to a copy of the rational line followed by a copy of the real line. We now generalize this idea.

Let Λ a set of ordinals less than λ and let Λ^c be its complement in λ . Let O_Λ be the set $\Lambda \times Q + \Lambda^c \times R$ where Q are the rational numbers and R are the real numbers. On O_Λ the order is defined as follows: $\langle \alpha, a \rangle < \langle \beta, b \rangle$ if either $\alpha < \beta$ (with respect to

the order of ordinals) or $\alpha = \beta$ and $a < b$ (with respect to the ordering of R). The reader can prove as an exercise that this ordering is dense and that O_{Λ_1} and O_{Λ_2} are isomorphic only when $\Lambda_1 = \Lambda_2$. The cardinality of O_Λ is $|2^{\aleph_0} \lambda|$ so, this prove the claim for every λ larger than 2^{\aleph_0} . To extend the result to cardinals $\aleph_0 < \lambda < 2^{\aleph_0}$, we only need replace R with a smaller dense linear order. This exists by the downward Löwenheim-Skolem theorem that we prove in 3.3.2 below.

3.2 Elementary maps and partial isomorphisms

Structures may have small subsets that look alike. Partial maps between structures are used to compare small pieces of structures.

3.2.1 Elementary maps.

Let M and N be structures containing A and fix some $\bar{a} \subseteq M$ and $\bar{c} \subseteq N$. We write $M, \bar{a} \equiv_A N, \bar{c}$ if for all A -formulas we have

$$* \quad M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{c}).$$

When M and N are the same model and this is clear from the context then we write $\bar{a} \equiv_A \bar{c}$. We omit mentioning A if this is empty. These notions extend to infinite tuples: we write $M, \bar{a} \equiv_A N, \bar{c}$ with the obvious meaning.

We say that $F : M \rightarrow N$ is an **A -elementary map** if for every tuple \bar{a} in the domain of definition of F we have $M, \bar{a} \equiv_A N, F\bar{a}$. A technical remark: though we do not require here that A belongs to the domain of definition of the map there is always a unique A -elementary extension of $F : M \rightarrow N$ defined on $\text{dom } F + A$, namely $\text{id}_A + F : M \rightarrow N$.

A total elementary map $F : M \rightarrow N$ is called an **A -elementary embedding** of M into N . The reader can easily check that elementary maps are injective and that the class of A -elementary maps is closed under inverse and composition (whenever this is defined). EXERCISE

3.2.2 Partial isomorphisms

Let M and N be structures containing A and fix some $\bar{a} \subseteq M$ and $\bar{b} \subseteq N$. If $(*)$ above holds for every quantifier-free formula $\varphi(\bar{x})$ then we write $M, \bar{a} \text{ qf} \equiv_A N, \bar{b}$. Again M, N , and A can be omitted with the same conventions as above. We say that $F : M \rightarrow N$ is a **partial A -isomorphism** if $M, \bar{a} \text{ qf} \equiv_A N, F\bar{a}$ for every tuple \bar{a} in the domain of definition of F . Clearly every A -elementary map is a partial A -isomorphism.

The name *partial isomorphism* is unfortunate: *partial* usually refers to the domain

of definition while here it actually refers also to the range. A partial isomorphism which is both total and surjective is clearly an isomorphism (so, in particular, an elementary map). A total partial isomorphism $F : M \rightarrow N$ is called an **A-embedding** of M into N . Again, it is easy to check that partial isomorphisms are injective and that the class of partial A -isomorphisms is closed under inverse and composition.

3.2.3 Generated substructures.

It is immediate that every partial isomorphism $F : M \rightarrow N$ has an extension $H : M \rightarrow N$ to a partial isomorphism defined on the substructure of M generated by $\text{dom } F$ and as range the substructure of N generated by the range of F . Moreover this extension is unique. In fact, every element c of the substructure generated by $\text{dom } F$ is of the form $t(\bar{a})$ for some term $t(\bar{x})$ and some $\bar{a} \subseteq \text{dom } F$. So we need only define $F(c)$ to be $t(F(\bar{a}))$ and check that, since F is partial isomorphism, the value of F on c is independent of the choice of \bar{a} and $t(\bar{x})$. The same is true for elementary maps. EXERCISE

3.2.4 Elementary maps and substructures.

It is immediate from the respective definitions that $M \preceq N$ if and only if the map $\text{id}_M : M \rightarrow N$ is elementary. It follows that if $F : M \rightarrow N$ is an elementary map and $K \preceq M$ then $F \upharpoonright K : K \rightarrow N$ is also an elementary map. The reader can prove as an exercise that the range of an elementary embedding $F : M \rightarrow N$ is an elementary substructure of N . (This last claim can be checked directly or one can apply the Tarski-Vaught test in 3.3.1 below.) EXERCISE

3.2.5 Automorphisms and elementary substructures

The reader may prove as an exercise the following useful observations. If $F : M \rightarrow N$ is an isomorphism of M and $K \preceq M$ then FK is (the domain of) an elementary substructure of N . If $F : M \rightarrow M$ is an automorphism that fixes $N \preceq M$ setwise, that is, when $FN = N$, then $F \upharpoonright N$ is an automorphism of N . (The notation has been fixed in 1.1.11 above). EXERCISE

3.2.6 Mapping formulas.

To any map $F : M \rightarrow N$ we associate an action on formulas. Namely, formulas over $\text{dom } F$ are mapped to formulas over $\text{rng } F$. This action is defined as follows. Every formula over M can be written in the form $\varphi(\bar{x} \bar{a})$, where $\varphi(\bar{x} \bar{y})$ is parameter-free and $\bar{a} \subseteq M$ (cf. Exercise 2.1.10 above). We define the image of the formula $\varphi(\bar{x} \bar{a})$, where $\varphi(\bar{x} \bar{y})$ is parameter-free, and $\bar{a} \subseteq \text{dom } F$ to be the formula $\varphi(\bar{x} F\bar{a})$. The image of $\varphi(\bar{x})$ under F is denoted by $F\varphi(\bar{x})$ (this is non-standard notation). A

similar notation is used for set of formulas. So the definition of elementary map above can be restated as follows. The map $F : M \rightarrow N$ is elementary if and only if F preserves truth of sentences over $\text{dom } F$, that is, for every sentence over $\text{dom } F$

$$M \models \varphi \Leftrightarrow N \models F\varphi.$$

The same is true for partial isomorphisms if we restrict to quantifier-free sentences. The same characterization is true when localized at some set of parameters A if we replace F with $F + \text{id}_A$ (and require that F is compatible with id_A).

3.2.7 Isomorphisms are elementary maps.

The following is a rephrasing of Proposition 2.3.3 above. We state it, without a proof, only for emphasis. It is important to stress that the claim is about isomorphisms and not about *partial* isomorphisms: the theorem is false if we relax the hypotheses of totality and/or surjectivity of $F : M \rightarrow N$.

PROPOSITION Every isomorphism $F : M \rightarrow N$ is an elementary map.

3.2.8 Elementary maps versus partial isomorphisms.

Let us contrast the notion of elementary map with that of partial isomorphism. From the definitions it is clear that all elementary maps are partial isomorphisms. The converse is true only in a weak sense: when $F : M \rightarrow N$ is a partial isomorphism then the structures generated by $\text{dom } F$ and $\text{rng } F$ are isomorphic; so, by the theorem above, they are elementary equivalent. But the truth in the substructures generated by $\text{dom } F$ and $\text{rng } F$ may be only loosely related to the truth in M and N unless these are elementary substructures. The following proposition shows that this is exactly what makes the difference (in the next chapters we see that the converse of this proposition holds if M and N are saturated enough).

PROPOSITION Suppose that $F : M \rightarrow N$ is a partial isomorphism and $\text{dom } F$ and $\text{rng } F$ are elementary substructures of M , respectively, N . Then $F : M \rightarrow N$ is an elementary map.

PROOF Since $F : \text{dom } F \rightarrow \text{rng } F$ is an isomorphism, it preserves truth between these structures. Since both $\text{dom } F$ and $\text{rng } F$ are elementary substructures, the truth in $\text{dom } F$ respectively in $\text{rng } F$ coincides with the truth in M , respectively, N . So $F : M \rightarrow N$ is elementary. The details are left to the reader.

3.2.9 The elementary chain lemma for maps.

Let $F : M \rightarrow N$ and $F' : M' \rightarrow N'$ be two elementary maps. We say that $F' : M' \rightarrow N'$ is an **elementary extension** of $F : M \rightarrow N$ if $M \preceq M'$, $N \preceq N'$, and F' extends F . A chain of elementary maps is defined in the obvious way. The proof of the following proposition is left to the reader.

PROPOSITION Let $F_\alpha : M_\alpha \rightarrow N_\alpha$ be a elementary chain of maps, that is, a sequence of elementary maps where M_α and N_α are elementary chains of structures and F_α is a chain of functions. Then the limit of the chain is an elementary map that extends all maps in the chain.

3.2.10 The diagram of a structure.

We associate to the theories $\text{Th}_M M$ and $_{\text{qf}}\text{Th}_M M$ some set of formulas. These are, roughly, obtained by identifying the elements of M with variables. Let \bar{a} be a tuple with range M and let \bar{x} have the same length of \bar{a} . Let $A \subseteq M$, the set $p(\bar{x})$ of A -formulas $\varphi(\bar{x})$ that is realized (that is, satisfied) by \bar{a} is called the **elementary diagram of M over A** . The set $p(\bar{x})$ of quantifier-free formulas $\varphi(\bar{x})$ that is realized by \bar{a} is called the **quantifier-free diagram of M over A** . When A is empty we omit it from the terminology.

3.2.11 The diagram lemma.

The following proposition is the motivation behind the definition of the diagram.

PROPOSITION If N realizes the elementary diagram of M then there is an elementary embedding of M into N . If N realizes the quantifier-free diagram of M then there is an embedding of M into N .

PROOF The proof of the two claims is identical. So we prove only the first one. Let $p(\bar{x})$ be the elementary diagram of M and let \bar{a} be the enumeration of M that realizes $p(\bar{x})$. Let \bar{c} realize $p(\bar{x})$ in N and let $F : M \rightarrow N$ map \bar{a} to \bar{c} . We check that this is an elementary embedding. Observe first that $\bar{a} \mapsto \bar{c}$ is indeed a function (this check is left to the reader). An arbitrary M -sentence can be written as $\varphi(\bar{a})$ where $\varphi(\bar{x})$ is a parameter-free. So if $\varphi(\bar{a})$ holds in M then $\varphi(\bar{x})$ is in the elementary diagram so $\varphi(\bar{c})$ holds in N .

3.2.12 An application.

We apply the diagram lemma above to show that any pair of finite elementary equivalent structures are isomorphic.

PROPOSITION Let M and N be finite structures and suppose that $M \equiv N$. Then M and N are isomorphic.

PROOF Assuming the hypothesis, then the sentence $\exists^{=n} x (x = x)$, that says that there are n elements in the domain, holds in M if and only if it holds in N . So M and N have the same cardinality and it is sufficient that we can embed one into the other. Let $p(\bar{x})$ be the quantifier-free diagram of M . So if we show that $p(\bar{x})$ is realized in N , the proposition follows by the diagram lemma. Suppose not for a contradiction. For every $\bar{a} \subseteq N$ choose a formula $\varphi(\bar{x})$ in $p(\bar{x})$ that is not realized by

\bar{a} and let $\psi(\bar{x})$ be the conjunction of all these (finitely many) formulas. Then $\psi(\bar{x})$ is inconsistent in N . Then $N \not\models \exists \bar{x} \psi(\bar{x})$, but $\psi(\bar{x})$ belongs to $p(\bar{x})$, so $M \models \exists \bar{x} \psi(\bar{x})$ a contradiction.

3.3 The downward Löwenheim-Skolem theorem

In general, there is no natural notion of elementary substructure of N generated by some set $A \subseteq N$. The best we can obtain is to prove the existence of *some* elementary substructure. This is the so-called downward Löwenheim-Skolem theorem: given $A \subseteq N$ we there is a model M of cardinality $|LAN_0|$ such that $A \subseteq M \preceq N$. Though this theorem does not give us any information about M (but for an upper bound on the cardinality) the proof introduces a very flexible technique. In the next chapters this technique will be adapted to more complex constructions. So it is useful to dedicate some attention to the procedure in its simplest setting: many constructions in the next chapters tacitly refer to 3.3.3 below for the details of the procedure.

3.3.1 The Tarski-Vaught test.

To prove that $M \preceq N$ requires the verification, for every M -sentence, that φ holds in M if and only if it holds in N . In some case this can be proved by induction on the complexity of φ . Though induction requires the verification of many cases, only one of these (or even less: just one direction of one induction case) is non-trivial. This is made precise by the following proposition. Note that in 2 below all parameters are in M but truth is evaluated (both times) in N .

PROPOSITION Let M be a substructure of N . The following are equivalent:

- 1 $M \preceq N$; and
- 2 for every M -formula $\varphi(x)$ such that $N \models \exists x \varphi(x)$ there is an $a \in M$ such that $N \models \varphi(a)$.

PROOF The proof of 1 \Rightarrow 2 is straightforward: if $\varphi(x)$ be an M -formula, $M \preceq N$, and $N \models \exists x \varphi(x)$ then, by elementarity, $M \models \exists x \varphi(x)$ so, by the definition of truth, $M \models \varphi(a)$ for some $a \in M$. Then, again by elementarity, $N \models \varphi(a)$. To prove the implication 2 \Rightarrow 1 we prove that for every parameter-free formula $\varphi(\bar{x})$ and every $\bar{a} \in M$ we have $M \models \varphi(\bar{a})$ if and only if $N \models \varphi(\bar{a})$. By Exercise 2.1.10 above, we obtain $M \preceq N$. We proceed by induction on the syntax of $\varphi(\bar{x})$. The atomic case is clear because M is a substructure of N . Induction for Boolean connectives is straightforward. The significant part of the proof is induction for existential quantification: suppose $N \models \exists x \varphi(\bar{a} x)$ for some $\bar{a} \subseteq M$; so, by hypothesis $N \models \varphi(\bar{a} b)$ for some $b \in M$. Now we apply the induction hypothesis to conclude that $M \models \varphi(\bar{a} b)$ and hence $M \models \exists x \varphi(\bar{a} x)$. The converse implication: if $M \models \exists x \varphi(\bar{a} x)$

then $N \models \exists x\varphi(\bar{a}x)$, follows directly from the induction hypothesis.

3.3.2 The downward Löwenheim-Skolem theorem.

Given $A \subseteq N$ we now construct a model M such that $A \subseteq M \preceq N$. So we must ensure $M \models \varphi$ for every M -sentence φ true in N . This seems to exclude the possibility of a direct construction for the reason that we cannot evaluate the truth in M when M is yet to be constructed. Fortunately, the Tarski-Vaught test replace this by a simpler requirement: every formula over M consistent in N is satisfied by some element of M . Since this does not mention truth in M but only truth in N , it is easier to check.

PROPOSITION Let N be an infinite structure and let $A \subseteq N$. There is an M of cardinality at most $|LAN_0|$ such that $A \subseteq M \preceq N$.

PROOF The Tarski-Vaught test suggests a method to construct M : add to A a witness of each consistent A -formula. The only difficulty is that the number of A -formulas increases as we keep adding elements to A . That is, if the set A_1 contains a witness for each A -formula, there is still no guarantee that all A_1 -formulas have a witness in A_1 . So we need to repeat the procedure to get a new set A_2 that contains a witness for each A_1 -formula. Iterating, we let A_{i+1} contain a witness for each A_i -formula. If we let M be the union of this chain for $i \in \omega$, then we are done: each M -formula has a finite number of parameters, so it is an A_i -formula for some sufficiently large i , so it has a solution in $A_{i+1} \subseteq M$.

3.3.3 Adding one element at the time.

In the proof above to move from A_i to A_{i+1} we add infinitely many elements. In the chapters below many proofs require that the elements of the model M satisfy some given properties. Then to better control the construction it is preferable to enumerate the elements of M one-by-one. This requires the discussion of a few technical details (we shall not pay much attention to these details in the following chapters; there we tacitly refer to this paragraph).

Let λ be a cardinal. We are going to use a well-ordering of all pairs $\langle \beta, \gamma \rangle$ of ordinals $< \lambda$. We can assume that this well-ordering has length λ and that the α -th pair $\langle \beta, \gamma \rangle$ in the well-ordering is such that $\beta, \gamma < \alpha$. Now we reprove the theorem above.

PROOF The required model M is constructed as limit of the chain: $A_0 = A$; $A_{\alpha+1} = A_\alpha + a_\alpha$; and union at limit stages. The chain has length $|LAN_0|$ and each set in the chain has cardinality at most $|LAN_0|$. The elements $a_\alpha \in N$ are obtained as follows. Along with the construction of A_α we fix, at each stage α , an enumeration $\psi_\gamma^\alpha(x)$ of all consistent A_α -formulas. There are $|LAN_0|$ formulas over A_α . Then a_α is chosen arbitrarily among the witnesses of $\psi_\gamma^\alpha(x)$ where $\langle \beta, \gamma \rangle$ is the α -th pair in the

well-ordering above. Clearly, at stage $|LAN_0|$ all M -formulas have been considered.

3.3.4 An exercise.

Let $M \preceq N$ and let $A \subseteq N$. Prove that there is a model K of cardinality $|LAN_0|$ such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$.

EXERCISE

Chapter 4

Compactness

In this chapter we prove the compactness theorem and use it to construct monster models (or universal domains).

How not to read this chapter.

The first section contains important definitions and there is no way to avoid them. On the contrary, the proof of the compactness theorem uses a method (so-called method of the Henkin constants) that we will not encounter in the rest of these notes. This proof may be skipped at a first reading. The main result of Section 4.3 is the existence of large saturated structures (also called monster models or universal domains). Our suggestion is to assume on trust the existence of saturated models and start working with them. Learning how to use saturation is by far more important than the actual construction of a saturated model: the drills in Section 4.4 are essential.

Notation and global assumptions.

The signature L is fixed, so **structure** below stands for structure of signature L . We use the symbols M, N, K , etc. for structures and A, B, C , etc. for sets of parameters (i.e. subsets of some structure).

4.1 Consistency

It is possible to read the word *model* in this section as synonym of *structure of signature* L and we suggest to do so at a first reading. But it is good to note that all the notions introduced in this section make sense also when models are restricted to range over some fixed class of structures. Eventually we need to relativize the concepts introduced here to different contexts, e.g. the class of elementary substructures

of a monster model, or the class of structures that models some theory T .

4.1.1 Realizations.

Let $p(\bar{x})$ be a set of formulas. Recall that \bar{a} is a **realization of p in M** if M contains \bar{a} and the parameters of p and $M \models p(\bar{a})$. We can also say: \bar{a} **realizes (or satisfies) p in M** . We say that **M realizes p** or that **p is consistent in M** if there is a realization of p in M . We say that p is **consistent** if it is consistent in some model.

4.1.2 Consistency and finite consistency.

With the same notation as the paragraph above. A consistent set of formulas p is called a **type**. A consistent set of sentences T is called a **theory**. We say that the type p or the theory T is **finitely consistent** if every finite subset is consistent (possibly each finite subset in a different model). If all finite subsets of p are consistent in the same model M (possibly each realized by a different assignment to the free variables of p) then we say that p is finitely consistent **in M** .

4.1.3 Consistency, isomorphisms, and elementarity.

Let $F : M \rightarrow N$ be an elementary map and suppose that $\text{dom } F$ contains the parameters of p . Recall that we write Fp for the image of p under F , that is, we replace the parameters occurring in p with their image under F . Observe that the consistency of an infinite set of formulas p in M does not guarantee the consistency of Fp in N . For an easy counter-example let M be the rational numbers with their natural order, N the substructure the substructure of M with domain the open interval $(0, 1)$. Let $F : M \rightarrow N$ be the identity on the set $\{n^{-1} : n \in \omega\}$ and undefined elsewhere. The argument used in Paragraph 3.1.7 can be adapted to show that $F : M \rightarrow N$ is an elementary map. Let $p(x)$ be the set of formulas $\{x < n^{-1} : n \in \omega\}$. Clearly $p(x)$ is consistent in M and inconsistent in N .

This constants with the fact that *finite* consistency is preserved by elementary maps. In fact, when p is finite, the conjunction the formulas in p is a first order formula φ . The conjunction of the formulas in Fp is equivalent to $F\varphi$. A formula $\varphi(\bar{x})$ is consistent if $\exists \bar{x}\varphi(\bar{x})$ is true, so elementary maps preserve consistency of formulas.

4.1.4 Logical consequences.

Let $p(\bar{x})$ be a set of formulas and let $\varphi(\bar{x})$ be a formula. We write $p \vdash \varphi$ if $M \models \varphi(\bar{a})$ for every model M and every tuple \bar{a} realizing p in M . In other words if in every model M we have $p(\bar{M}) \subseteq \varphi(\bar{M})$. In words we say that φ is a **consequence of p** . Note that when \bar{x} contains a variable that does not actually occur in p , say $\bar{x} = \bar{y}\bar{z}$

where \bar{z} does not occur in p , then $p \vdash \varphi$ is equivalent to $p \vdash \forall \bar{z} \varphi$. In particular $T \vdash \varphi$, where T is a theory, means that $\forall \bar{x} \varphi(\bar{x})$ holds in every model of T . Below we write ${}_{\text{qf}}\mathcal{P}(\bar{z})$ for the set of quantifier-free formulas with free variables among \bar{z} that are consequences of p . Analogously, ${}_{\text{qf}}\mathcal{T}$ is the set of quantifier-free sentences that are consequence of T . These sets may be empty, in particular, when L contains no constants, ${}_{\text{qf}}\mathcal{T}$ is always empty. We say that q is an **axiomatization of p** if q is a subset of p and every formula in p is a consequence of q .

4.1.5 Completeness.

We say that p **decides φ** if exactly one of $p + \varphi$ and $p + \neg\varphi$ is consistent, that is, either $p \vdash \varphi$ or $p \vdash \neg\varphi$ but not both. We say that p is **complete over A** if p decides every A -formula φ . When the set of parameters A is not specified then we assume that it is the set of parameters occurring in p .

4.1.6 Completion of a set of formulas.

For the proof of the compactness theorem in the next chapter we need the following lemma.

PROPOSITION Let p be a finitely consistent set of formulas. Then p is contained in a complete and finitely consistent set of formulas q . Moreover, we may require that q has cardinality at most $|Lp\aleph_0|$.

PROOF Let \bar{x} be a tuple of variables containing the free variables of p and of length $\leq |p|$. Fix an enumeration φ_α of the formulas whose parameters occur in p and free variables in \bar{x} . We can assume that this enumeration has length $|Lp\aleph_0|$. The required set q is constructed as limit of the chain q_α . The chain starts with p . At limit stages we take the union: it is clear that finite consistency is preserved under taking the union of a chain. At stage $\alpha + 1$, define $q_{\alpha+1}$ to be $q_\alpha + \varphi_\alpha$ or, if this is not finitely consistent, $q_\alpha + \neg\varphi_\alpha$. It is immediate to check that if q_α is finitely consistent then it is finitely consistent either with φ_α or with $\neg\varphi_\alpha$. Clearly the construction yields a complete set.

4.2 Compactness

We prove first the compactness theorem for sets of quantifier-free formulas. Then we show that it is possible to reduce the consistency of a set of arbitrary formulas to the consistency of a set of quantifier-free formulas.

4.2.1 The compactness theorem (the quantifier-free case).

Here we prove the quantifier-free case of the compactness theorem, this will be generalized below to arbitrary formulas.

PROPOSITION Let p be a finitely consistent set of quantifier-free formulas. Then p is consistent and there is structure N realizing p of cardinality at most $|Lp\aleph_0|$.

PROOF Let A be a set containing all the parameters of p . First we prove the theorem under the following two assumptions: **1** p is complete for quantifier-free A -formulas; and **2** for every term t occurring in p there is a variable y such that $t = y$ is in p . Let V be the set of variables occurring in p . For t and s two atomic A -terms (that is elements of AV) we write $t \sim s$ if and only if $t = s \in p$. From (1) it follows immediately that \sim is an equivalence relation. By (2) every equivalence class contains a variable. Let $W \subseteq V$ be such that W contain exactly one element of each equivalence class. The set W is the domain of the structure N . Now we define the interpretation in N of the functions of L . For every function symbol f and every $\bar{x} \subseteq W$ we define f^N to be the unique element $y \in W$ in the equivalence class of $f\bar{x}$ (so $f\bar{x} = y$ is in p). Now, for every relation symbol r and every $\bar{x} \subseteq W$ we stipulate that $\bar{x} \in r^N$ if and only if $r\bar{x} \in p$. Literally speaking N does not contain A , but we will identify a with the unique variable in W in the equivalence class of a . It is also clear that p is realized in N by the assignment that maps the variable x to the unique $y \in W$ such that $x \sim y$. Finally, observe that the cardinality of V is clearly less than $|p|$, so N also has the cardinality of $|Lp\aleph_0|$. This completes the proof of the theorem from assumption (1) and (2).

To finish the proof we need to show that any finitely consistent set p can be extended to a set meeting these two requirements. We first observe that every finitely consistent set p is contained in a finitely consistent set satisfying (2). This is obvious: fix for each term t in p a fresh variable y and add the formula $t = y$. The new set is clearly finitely consistent (as for each term we use a distinct fresh variable). The lemma above yields a finitely consistent set containing p and satisfying (1). Unfortunately the two constructions need not a priori be compatible: one may destroy what the other has achieved. So we define a chain p_i of length ω ; the even stage of the chain satisfy (1) and the odd stages satisfy (2). Clearly the union of the chain meets both (1) and (2). Since we may also require that the cardinality of p_{i+1} does not exceed $|Lp_i\aleph_0|$, the union of the chain has cardinality $|Lp\aleph_0|$.

4.2.2 Fullness.

We need the following technical notion, the terminology is non-standard. Let p be a set of formulas. We say that p is **full** if:

- 1** p is complete; and
- 2** if $\exists y \varphi(\bar{x}y)$ is in p then for some variable z the formula $\varphi(\bar{x}z)$ is in p .

4.2.3 A lemma.

When a set of formulas p is full, then we are able to find the description of a model realizing p hidden in the quantifier-free formulas of p . Precisely, we have the following fact.

PROPOSITION Let $p(\bar{x})$ be full set of formulas. Let suppose \bar{a} realizes $_{\text{qf}}p(\bar{x})$. Then the range of \bar{a} is the domain of a model M and \bar{a} realizes $p(\bar{x})$ in M .

PROOF First check that the elements occurring in the tuple \bar{a} form the domain of a structure M . It suffices to prove that $t^M(\bar{a})$ is an element that occurs in \bar{a} for every term $t(\bar{x})$. By the completeness and finite consistency of $p(\bar{x})$ some formula of the form $\exists y (t(\bar{x}) = y)$ is in $p(\bar{x})$ (in fact, its negation is inconsistent). By fullness, the quantifier-free formula $t(\bar{x}) = z$ is in $p(\bar{x})$ for some z . So it is realized by \bar{a} . This proves the claim. Now we prove that \bar{a} realizes $p(\bar{x})$ in M by induction on the syntax of the formulas in $p(\bar{x})$. For atomic formulas it is clear: \bar{a} is assumed to realize the quantifier-free formulas of $p(\bar{x})$. Induction for the connectives \wedge and \neg is straightforward. Suppose $\exists y \varphi(\bar{x}y)$ is in $p(\bar{x})$, then for some variable z the formula $\varphi(\bar{x}z)$ is in $p(\bar{x})$. By induction hypothesis that \bar{a} satisfies $\varphi(\bar{x}z)$ in M . Then \bar{a} satisfies $\exists y \varphi(\bar{x}y)$ in M .

4.2.4 A lemma.

The realization of a set of arbitrary formulas p is obtained in three steps. First extend p to a set q which is full. Then by the quantifier-free compactness theorem we obtain an infinite tuple \bar{a} that realizes the quantifier-free formulas of q . Finally, by the lemma above we obtain that the range of the assignment \bar{a} is the model where q (hence p) is realized. We now prove that the first step in this procedure is always possible. We also give a bound on the cardinality of q ; this is interesting because it yields a bound on the cardinality of the model realizing p . This bound is easily seen to be optimal.

PROPOSITION Let p be a finitely consistent set of formulas. Then there is a full and finitely consistent set of formulas containing p . Moreover we can require that this set has cardinality at most $|Lp\aleph_0|$.

PROOF We use the same idea as in final part of the proof of 4.2.1 above (observe that the requirement 2 in the definition of fullness is just a strengthening of assumption 2 in that proof). We first observe that every finitely consistent set p is contained in a finitely consistent set satisfying the second claim of the definition of fullness. This is obvious: fix for each formula $\exists y \varphi(\bar{x}y)$ in p a fresh variable z and add the formula $\varphi(\bar{x}z)$ to p . The new set is clearly finitely consistent (as for each formula we use a distinct fresh variable). We define a chain p_i of length ω ; the even stage of the chain satisfy 1 and the odd stages satisfy 2. Clearly the union of the chain meets both 1 and 2. Since we may also require that the cardinality of p_{i+1} does not

exceed $|p_i L\aleph_0|$, the union of the chain has cardinality $|p L\aleph_0|$.

4.2.5 The compactness theorem.

Summarizing what is obtained by the two lemmas above, we finally get the compactness theorem.

PROPOSITION Let p be a finitely consistent set of formulas. Then p is consistent. Moreover there is a structure realizing p of cardinality at most $|Lp\aleph_0|$.

4.2.6 Realizations in elementary extensions.

Fix a model M . Let p be a set of M -formulas. Clearly p is finitely consistent in M then $\text{Th}_M M + p$ is finitely consistent. So we have the following.

PROPOSITION If p is finitely consistent in M then p is realized in some elementary extension of M .

4.2.7 The upward Löwenheim-Skolem theorem.

The following remarkable observations are direct consequences of the compactness theorem above. Let M be an infinite model. Then for every infinite tuple \bar{x} of variables the set $p(\bar{x})$ containing the formulas $x \neq y$ for every distinct $x, y \in \text{rng } \bar{x}$ is clearly finitely consistent in M . Every model realizing $p(\bar{x})$ has cardinality at least $|\text{rng } \bar{x}|$. So, from the proposition above it follows that every model M has an elementary extension of arbitrarily large cardinality.

4.2.8 An exercise

Let p be a set of formulas and let φ be a formula. Prove that $p \vdash \varphi$ if and only if $q \vdash \varphi$ for some finite $q \subseteq p$. **EXERCISE**

4.2.9 An exercise

Let $p(\bar{x} \bar{z})$ be a set of formulas closed under conjunction and let $\varphi(\bar{x})$ be a formula. Let $q = \{\exists \bar{z} \psi(\bar{x} \bar{z}) : \psi \in p\}$. Prove that $p \vdash \varphi$ if and only if $q \vdash \varphi$. **EXERCISE**

4.3 Monster models

Eventually in the next chapters we are going to fix a complete theory T and a large model of T , a so-called *monster model*. We will restrict the attention to elementary substructures of this large structure. We call this large structure a *universal domain* or simply a *the universe* of T . In this section we define saturation the notion that

we use to formalize our model theoretical intuition of *large*. We prove some easy facts about saturation; the discussion will be continued in Chapter 6.

4.3.1 Saturation.

A structure M is **saturated** if, for every set $A \subseteq M$ of cardinality $< |M|$, every set of A -formulas $p(x)$ that is finitely consistent in M is realized in M .

4.3.2 Saturation with many variables.

The definition of saturation given above requires that all formulas in $p(x)$ have only one free variable. But the following proposition shows that the number of free variables is not very relevant.

PROPOSITION Let M be a saturated structure. Let \bar{x} have length $\leq |M|$. Let $p(\bar{x})$ be a set of M -formulas over $< |M|$ parameters. Suppose also that $p(\bar{x})$ is finitely consistent in M . Then $p(\bar{x})$ is realized in M .

PROOF Without loss of generality we can assume that $p(\bar{x})$ is closed under conjunction. A realization \bar{a} is constructed by stages. Denote by a_α and x_α the α -th component of \bar{a} , respectively, \bar{x} . At stage α we assume a_β is defined for every $\beta < \alpha$ and define a_α . Let $p_\alpha(\bar{x}_\alpha)$ be obtained by substituting a_β for x_β in $p(\bar{x})$ for every $\beta < \alpha$. Assume as induction hypothesis that $p_\alpha(\bar{x}_\alpha)$ is consistent in M . Let $q_\alpha(x_\alpha)$ contain formulas of the form $\exists \bar{x} \varphi(x_\alpha \bar{x})$ where $\varphi(x_\alpha \bar{x})$ is in $p_\alpha(\bar{x}_\alpha)$ and x_α does not occur in \bar{x} . Clearly $q_\alpha(x_\alpha)$ is finitely consistent and, by saturation, there is a realization $a_\alpha \in M$. We check that $p_{\alpha+1}(\bar{x}_{\alpha+1})$ is finitely consistent in M . Since this set is closed under conjunction, it suffices to show that every formula is consistent in M . Formulas in $p_{\alpha+1}(\bar{x}_{\alpha+1})$ have the form $\varphi(a_\alpha \bar{x})$ where $\varphi(x_\alpha \bar{x})$ is in $p_\alpha(\bar{x}_\alpha)$. By the choice of a_α we have that $\exists \bar{x} \varphi(a_\alpha \bar{x})$ is true in M . So $\exists x_\alpha \bar{x} \varphi(x_\alpha \bar{x})$ is true in M as required.

4.3.3 A remark.

In the proposition above the requirements on the number of parameters and on the number of free variables are necessary. In fact, for every infinite structure M the set $\{x \neq a : a \in M\}$ is finitely consistent in M but clearly cannot be realized in M itself. Also the set $\{x \neq y : x, y \in V\}$ is finitely consistent in every infinite structure M but it cannot be realized in M when $|M| < |V|$.

4.3.4 Existence of saturated structures

Here we prove that large saturated models exist (if we assume the existence of inaccessible cardinals).

PROPOSITION Let T be a complete theory with an infinite model. Let κ be an inaccessible cardinal larger than $|T|$. There is a saturated model of T of cardinality κ .

PROOF The structure M is the union of an elementary chain M_α of length κ . The chain may start with any model of T of cardinality less than κ . Since the chain is elementary, the union will be a model of T . The chain is easily seen to be proper (the set of formulas $x \neq a$ for all $a \in M_\alpha$ will be realized at some stages) so its limit has actually cardinality κ . We use a well-ordering of all pairs $\langle \beta, \gamma \rangle$ of ordinals $< \kappa$. We can assume that this well-ordering has length κ and that the α -th pair $\langle \beta, \gamma \rangle$ in the well-ordering is such that $\beta, \gamma < \alpha$. Fix a variable x . At each stage α of the construction we fix a well-ordering of all finitely consistent set of M_α -formulas $p(x)$. By the inaccessibility of κ this well-ordering has length $< \kappa$. Suppose M_α has been constructed, then $M_{\alpha+1}$ is constructed as follows. Let $\langle \beta, \gamma \rangle$ be the α -th pair of ordinals $< \kappa$ and let $p(x)$ be the β -th set of M_γ -formulas (in the well-ordering above). This set is finitely consistent in M_γ and, by elementarity, in M_α . Let $M_{\alpha+1}$ be an elementary superstructure of M_α of cardinality less than κ where $p(x)$ is realized (this exists by 4.2.6 above). At limit stages we take the union of all previous stages: this we can do by the elementary chain lemma. This completes the definition of M . We need to show that M realizes every set of M -formulas of cardinality smaller than κ which is finitely satisfied in M . Let $p(x)$ be a set of M -formulas of cardinality smaller than κ that is finitely consistent in M but not realized in M . By the regularity of κ there is an $\gamma < \kappa$ such that all parameters of $p(x)$ are contained in M_γ . So $p(x)$ appear as, say, β -th set in the enumeration of M_γ -formulas. So, if $\langle \beta, \gamma \rangle$ is the α -th pair of ordinals $< \kappa$, then $M_{\alpha+1}$ realizes $p(x)$. By elementarity, the same element a that realizes $p(x)$ in M_α realizes $p(x)$ in M .

4.3.5 Topological interpretation of saturation.

The saturation of M may be interpreted as the claim that certain topologies on M are compact. Namely, to every set $A \subseteq M$ of cardinality $< |M|$ we associate the topology whose closed sets are arbitrary intersections of A -definable sets. It is clear that M is saturated if and only if all these topologies are compact.

4.3.6 The upward Löwenheim-Skolem theorem (inside a structure).

Include the following easy observation for future reference. Let M be a substructure of U . Can we find a structure N such that $M \preceq N \preceq_{\text{qf}} U$? The answer is affirmative when U is a saturated structure.

PROPOSITION Let U be a saturated structure and let $M \preceq_{\text{qf}} U$. Then for every $\lambda \leq |U|$ there is a structure N of cardinality λ such that $M \preceq N \preceq_{\text{qf}} U$. If λ is inaccessible then we can also require N is saturated.

PROOF By the upward Löwenheim-Skolem theorem there is a model K of cardinality λ such that $M \preceq K$. By the theorem above we can also require that K is saturated. Let $p(\bar{x})$ be the quantifier-free diagram of K over M . Since $p(\bar{x})$ is finitely consistent in K then, by elementarity, it is also finitely consistent in M and, since $M \preceq_{\text{qf}} U$, it is finitely consistent in U . So U realizes $p(\bar{x})$. By the diagram lemma, the range of this realization is a model $N \subseteq U$ isomorphic over M to K . So N models $\text{Th}_M M$, hence $M \preceq N$.

4.3.7 Examples and non-examples.

Saturated structures are rare in nature, so it is easier to give non-examples. Let L contain only the symbol $<$. Let R be the real numbers with their natural ordering. It is immediate to see that R is *not* saturated: consider the set of formulas $\{n < x : n \in \omega\}$. Let K be an infinite field. A vector space N over K of finite dimensions is *not* saturated: consider the set of formulas $\{x \neq t(\bar{a}) : t(\bar{z}) \text{ parameter free term}\}$, where \bar{a} is a base of N .

On the other side any vector space of dimension at least $|K|$ is saturated. We will not prove this for the moment, see Chapter 5 below.

4.3.8 Examples and non-examples (continued).

This example shows that saturated models may have large cardinality. Let L contain for every $i \in \omega$ a unary relation symbol r_i . Let T be a theory containing all the sentences of the form

$$\exists x \left[\bigwedge_{i \in I} r_i(x) \wedge \bigwedge_{i \in J} \neg r_i(x) \right],$$

where I and J are finite disjoint subsets of ω . A model M of T is obtained taking ${}^\omega 2$, the set of infinite sequences of 0's and 1's, as domain (that is, the set of total maps from ω to $\{0, 1\}$). The relations r_i are interpreted with the set of those $s \in {}^\omega 2$ such that $s(i) = 0$. It is easy to see that every saturated elementary extension of M has to contain an isomorphic copy of this model. The model M realizes all parameter-free types (we have not the tools to prove this yet), still, is not saturated. In fact, let $o \in {}^\omega 2$ be the sequence that is constantly 0. The set of formulas

$$x \neq o + \left\{ r_i(x) : i \in \omega \right\}$$

is finitely consistent in M but it is not realized by any element of ${}^\omega 2$.

4.3.9 Examples and non-examples (continued).

The definition of saturation does not mention the cardinality of $p(x)$ but only the cardinality of the parameters occurring in $p(x)$. This may be relevant when dealing with structures of cardinality $|L\aleph_0|$ (an intuition about saturation is better obtained

looking to examples of larger cardinality). Let R be like in 4.3.7 and let Q be the substructure of R with as domain the rational numbers. Then argument in 4.3.7 cannot be used to prove that Q is not saturated because the set of parameters in $\{n < x : n \in \omega\}$ has now the same cardinality as Q . We will see that Q is indeed a saturated structure.

Now, let L^+ is an expansion of L with infinitely many constants and expand Q to a structure Q^+ that interprets these constants in the integers $Z \subseteq Q$. The type $\{n < x : n \in \omega\}$ witnesses that Q^+ is not saturated.

4.3.10 Examples and non-examples (continued).

Let L contain the symbol $<$. Let Q be the rational numbers with their natural orderings. Let N be an uncountable saturated structure that extends Q elementarily. One can prove as an exercise that N contains elements that are larger than (and also smaller than) any element of Q . Moreover every element of Q is isolated by an interval of N , that is, for every $a \in Q$ there $b, c \in N$ such that a is the unique element of Q such that $b < a < c$.

EXERCISE

4.4 Daily life in a saturated universe

The few simple (non-)examples above are certainly not sufficient to give an intuition about saturation. Typically, saturated structures are artifacts that we use for mere convenience, the few natural examples we met above are just exceptions and it would be misleading to base our intuition on them. Intuition is better gained by proving easy facts about saturated structures.

4.4.1 Notation.

Below we propose a series of drills; they all assume the following notation. Let \mathcal{U} be a saturated structure of cardinality κ larger than $|T|$ ($= |L\aleph_0|$). Cardinals that are $< \kappa$ are called *small*. We say *formula* with parameters in \mathcal{U} and *type* for a set of formulas of small cardinality that is consistent in \mathcal{U} . We say *model* for elementary substructure of \mathcal{U} .

4.4.2 Lemma about first-order definability of type-definable sets.

The following straightforward lemma is quite useful to obtain easy definability results (especially its corollary (a) below).

PROPOSITION Let p and q be two types closed under conjunction and such that $p + q$ is inconsistent. Then there are $\varphi \in p$ and $\psi \in q$ such that $\varphi + \psi$ is inconsistent.

4.4.3 Corollaries.

With the notation fixed in 4.4.1 above. Let $p(\bar{x})$ and $q(\bar{x})$ be types closed under conjunctions and such that $p + q$ is inconsistent (in other words, $p(\bar{\mathcal{U}})$ and $q(\bar{\mathcal{U}})$ are disjoint sets). The reader can prove the following as an exercise:

a if $\bar{\mathcal{U}} = p(\bar{\mathcal{U}}) \cup q(\bar{\mathcal{U}})$ then $\varphi(\bar{\mathcal{U}}) = p(\bar{\mathcal{U}})$ and $\psi(\bar{\mathcal{U}}) = q(\bar{\mathcal{U}})$ for some formulas $\varphi \in p$ and $\psi \in q$.

Moreover, for every formula $\vartheta(\bar{x})$:

b if $p(\bar{\mathcal{U}}) \subseteq \vartheta(\bar{\mathcal{U}})$ then $\varphi(\bar{\mathcal{U}}) \subseteq \vartheta(\bar{\mathcal{U}})$ for some formula $\varphi \in p$; and

c if $p(\bar{\mathcal{U}}) = \vartheta(\bar{\mathcal{U}})$ then $\varphi(\bar{\mathcal{U}}) = p(\bar{\mathcal{U}})$ for some formula $\varphi \in p$. EXERCISE

4.4.4 A counter-example.

It is easy to see that the proposition above does not hold when \mathcal{U} is not saturated. Take for instance the structure of real numbers R with in the signature only the usual order relation. Let $p(x)$ contain the formulas $x < n^{-1}$ for every $n \in \omega$ and let $\vartheta(x)$ be the formula $x > 0$. Then $p(R)$ and $\vartheta(R)$ are disjoint though $\vartheta(x) \wedge \psi(x)$ is consistent in R for every $\psi \in p$. Note that extending R to a saturated structure \mathcal{U} we see that $p(\mathcal{U})$ and $\vartheta(\mathcal{U})$ are not disjoint: their intersection contains so-called *infinitesimals*.

4.4.5 A remark.

The facts above are often used to deduce the first-order definability of some type-definable sets. For instance (b) tells us that if the complement of $p(\bar{\mathcal{U}})$ is type-definable then $p(\bar{\mathcal{U}})$ is definable. But note that (b) also claims that $p(\bar{\mathcal{U}})$ is definable by a formula in $p(\bar{x})$. This may be a useful piece of information e.g. it may inform us about the complexity or about the parameters necessary to define $p(\mathcal{U})$. And all this is independent of the complexity and the parameters of the type-definition of $\mathcal{U} \setminus p(\bar{\mathcal{U}})$. We will see this principle many times in action e.g. in Paragraph 5.1.6 and 12.1.3 below.

4.4.6 Small \rightarrow finite (for definable sets).

With the notation of 4.4.1 above. **a** Show that for every formula $\varphi(\bar{x})$ the set $\varphi(\bar{\mathcal{U}})$ is either finite or has cardinality κ . **b** The same conclusion holds for $p(\bar{\mathcal{U}})$, where $p(\bar{x})$ is a type. Show that if in (b) we replace $p(\bar{x})$ with a set of formulas with infinitely many variables $p(\bar{x})$ then the claim becomes false. Hint: L contains a unary relation r_n for every $n \in \omega$ and $r_n^{\mathcal{U}}$ are pairwise disjoint sets of two elements. EXERCISE

4.4.7 Small \rightarrow finite (for uniform families of definable sets).

With the notation of 4.4.1 above. Let $\varphi(\bar{x}\bar{y})$ be an arbitrary formula. Show that the family of definable sets $\{\varphi(\bar{\mathcal{U}}\bar{a}) : \bar{a} \subseteq \mathcal{U}\}$ is either finite or has cardinality κ . The same holds if $\varphi(\bar{x}\bar{y})$ is replaced with a type $p(\bar{x}\bar{y})$ but the proof is more difficult. EXERCISE

4.4.8 On the definability of finiteness.

With the notation of 4.4.1 above. The following exercise says that there is no other way of saying (with a first-order formula) when a definable set is finite then saying that it contains at most n elements (for some fixed n).

Show that for every formula $\varphi(\bar{x}\bar{y})$ the following are equivalent: **a** there is a formula $\psi(\bar{y})$ such that for every $\bar{a} \subseteq \mathcal{U}$ we have that $\varphi(\bar{\mathcal{U}}\bar{a})$ is finite if and only if $\psi(\bar{a})$ holds; and **b** there is an n such that for every $\bar{a} \subseteq \mathcal{U}$ either $\varphi(\bar{\mathcal{U}}\bar{a})$ is infinite or it has cardinality $< n$. EXERCISE

4.4.9 Minimal and saturated \rightarrow strongly minimal.

With the notation of 4.4.1 above. Here we ask to prove equivalent two ways to say that T is strongly minimal. Strongly minimal theories will be studied in Chapter 8.1 below.

Prove that the following are equivalent: **a** for every parameter-free formula $\varphi(\bar{z}x)$ there is an n such that

$$\forall \bar{z} \left[\exists^{<n} x \varphi(\bar{z}x) \vee \exists^{<n} x \neg \varphi(\bar{z}x) \right]$$

holds; and **2** for every formula $\varphi(x)$ either $\varphi(\mathcal{U})$ or $\neg \varphi(\mathcal{U})$ is a finite set. EXERCISE

4.4.10 An exercise.

With the notation of 4.4.1 above. **1** Let $\bar{a} \subseteq \mathcal{U}$ and let $A \subseteq \mathcal{U}$ be an arbitrary set of small cardinality. Prove that if for every $\bar{b} \subseteq \mathcal{U}$ such that $\bar{b} \equiv_A \bar{a}$ we have $\bar{a} = \bar{b}$, then there is an A -formula such that $\varphi(\bar{a})$ and $\exists^{=1} \bar{x} \varphi(\bar{x})$ hold in \mathcal{U} . **2** Let $p(\bar{x})$ be a type which is closed under conjunctions. Prove that if $p(\bar{\mathcal{U}})$ is finite then $p(\bar{x})$ contains a formula with finitely many solutions. EXERCISE

Chapter 5

Collapse of quantifiers

Quantifiers have more complex semantic than Boolean connectives. Fortunately, in some cases the definable sets can also be obtained without the use of quantifiers. We will prove this happens for the theories of *dense linear orders*, of *vector spaces*, of *algebraically closed fields*, and of *real closed fields*. These proofs always require some specific understanding of the theory in question, but they also contain some routine work. The few general facts that we present in this chapter (typically, simple applications of compactness) may considerably simplify this routine.

Notation and global assumptions.

The signature L is fixed: below **structure** stands for structure of signature L . Let \mathbf{S} be theory without finite models. We work in the class of models of S : below by **model** we understand model of S (though we may repeat it for emphasis). The symbols M, N , etc. denote models.

5.1 Elimination of quantifiers

If every definable set is a Boolean combination of sets defined by atomic formulas, then we say that the theory has elimination of quantifiers. When the atomic definable sets are simple enough, quantifier elimination may simplify the description of all definable sets.

5.1.1 Elimination of quantifiers.

We say that \mathbf{S} **admits (or has) elimination of quantifiers** if for every parameter-free formula $\varphi(\bar{x})$ there is an quantifier- and parameter-free formula $\psi(\bar{x})$ equivalent to $\varphi(\bar{x})$ modulo S , where **modulo S** means in every model of S .

5.1.2 A formal detail.

The tuple \bar{x} in the definition above may be empty. So elimination of quantifiers requires that ${}_{\text{qt}}S$ (the set of quantifier- and parameter-free consequences of S) is non-empty otherwise there would be nothing the sentence $\forall x(x = x)$ we could be equivalent to. We do not want that the absence of a quantifier-free sentences is the only reason for quantifier elimination to fail and for this reason we introduced 0-ary connective \perp .

5.1.3 Ad hoc elimination of quantifiers.

Observe that the complexity (in an informal sense) of the definable sets in a theory with elimination of quantifiers, depends essentially on the complexity of the language. When L is finite, and/or we have an insight on the sets definable by atomic formulas, then elimination of quantifiers is informative (though, note that the number of Boolean connectives needed to eliminate one single quantifier may be very large, so definable sets may still be rather complex). There are cases when quantifier elimination is not informative. For instance one can always enforce an ad hoc and purely formal quantifier elimination: given any language L one can define a new language L^+ that has a relation symbol $r_{\varphi(\bar{x})}$ for every parameter-free formula $\varphi(\bar{x})$ of L . Let S^+ be the theory axiomatized by S and $\forall \bar{x}[r_{\varphi(\bar{x})}(\bar{x}) \leftrightarrow \varphi(\bar{x})]$ for every parameter-free formula $\varphi(\bar{x})$ of L . The theory S^+ is known as the *Morleyization* or the *Morley expansion* of S . Note that S^+ is not a very essential expansion of S . In fact every model of S has a unique expansion to a model of S^+ and this expansion do not add any new definable set. We claim that S^+ eliminates quantifiers: see the exercise below. This elimination of quantifiers is uninformative: it is only a matter of notation. Still, it may be used sometimes to clean the exposition of some arguments.

5.1.4 An exercise

Prove that the theory S^+ defined above above eliminates quantifiers. (Note that S^+ obviously eliminates quantifiers from formulas in the language L . But we claim elimination of quantifiers in L^+ .) Observe that every model of S has an expansion to a model of S^+ and that from every model of S^+ we obtain a model of S by simply forgetting the interpretation of relations not in L . EXERCISE

5.1.5 Universal-existential axiomatization.

From the observation above it follows that if S has elimination of quantifiers then S has a universal-existential axiomatization (that is an axiomatization by sentences of the form $\forall \bar{x} \exists \bar{y} \varphi(\bar{x} \bar{y})$ with $\varphi(\bar{x} \bar{y})$ quantifier-free). EXERCISE

5.1.6 A test for quantifier-free definability in a saturated model.

In the next paragraph we present a test that decides when a formula has a quantifier-free equivalent this is applied below to decide when a theory has elimination of quantifiers. The simple line of the argument appears more neatly when we work in a saturated model so we present first this particular case.

PROPOSITION Let \mathcal{U} be a saturated model. For every parameter-free formula $\varphi(\bar{x})$ the following are equivalent: (truth, consistency, equivalence is evaluated in \mathcal{U} , all elements are from \mathcal{U} and we write $\bar{a} \equiv_{\text{qf}} \bar{c}$ when \bar{a} and \bar{c} satisfy the same parameter- and quantifier-free formulas in \mathcal{U} .)

- 1 $\varphi(\bar{x})$ is equivalent to a parameter- and quantifier-free formula;
- 2 for every \bar{a}, \bar{c} , if $\bar{a} \equiv_{\text{qf}} \bar{c}$ then $\varphi(\bar{a}) \leftrightarrow \varphi(\bar{c})$; and
- 3 for every complete parameter- and quantifier-free type $p(\bar{x})$ either $p(\bar{\mathcal{U}}) \subseteq \varphi(\bar{\mathcal{U}})$ or $p(\bar{\mathcal{U}}) \subseteq \neg\varphi(\bar{\mathcal{U}})$.

PROOF Only the implication $3 \Rightarrow 1$ needs a proof. From 3 we have that

$$\varphi(\bar{\mathcal{U}}) = \bigcup_{p(\bar{\mathcal{U}}) \subseteq \varphi(\bar{\mathcal{U}})} p(\bar{\mathcal{U}}).$$

where $p(\bar{x})$ above range over the complete parameter- and quantifier-free types. By saturation and Exercise 4.4.2 above, $p(\bar{\mathcal{U}}) \subseteq \varphi(\bar{\mathcal{U}})$ is equivalent to $\psi(\bar{\mathcal{U}}) \subseteq \varphi(\bar{\mathcal{U}})$ for some $\psi(\bar{x})$ in $p(\bar{x})$. So we obtain

$$\varphi(\bar{\mathcal{U}}) = \bigcup_{\psi(\bar{\mathcal{U}}) \subseteq \varphi(\bar{\mathcal{U}})} \psi(\bar{\mathcal{U}}).$$

where $\psi(\bar{x})$ range over the quantifier-free formulas. Complementing, we obtain that $\neg\varphi(\bar{\mathcal{U}})$ is definable by the type

$$q(\bar{x}) = \left\{ \neg\psi(\bar{x}) : \psi(\bar{x}) \rightarrow \varphi(\bar{x}) \right\}$$

Again by saturation, we obtain that a finite number of formulas in $q(\bar{x})$ suffices.

5.1.7 Tests for quantifier elimination modulo complete theories.

These are the criteria for elimination of quantifiers that we promised in the introduction; in the examples of Section 5.3 we mainly use the last one.

PROPOSITION Let S be a complete theory and fix \mathcal{U} , an arbitrary saturated model of S . The following are equivalent (again all elements range over \mathcal{U} and \mathcal{U} is the domain and codomain of all maps):

- 1 S has quantifier elimination;
- 2 for every \bar{a} and \bar{c} , if $\bar{a} \equiv_{\text{qf}} \bar{c}$ then $\bar{a} \equiv \bar{c}$;
- 3 every partial isomorphisms is an elementary map; and
- 4 every $\bar{a} b$ and \bar{c} , if $\bar{a} \equiv_{\text{qf}} \bar{c}$ then there is a d such that $\bar{a} b \equiv_{\text{qf}} \bar{c} d$.

PROOF Reason in \mathcal{U} (i.e. evaluate consistency and truth in \mathcal{U}). The equivalence $2 \Leftrightarrow 3$ is evident. The equivalence $1 \Leftrightarrow 2$ is also clear: if S has quantifier elimination,

then in every model the relation of elementary equivalence \equiv reduces to \equiv_{qf} . The converse follows from the proposition above and the completeness of S . To prove the implication $2 \Rightarrow 4$ assume $\bar{a} \equiv_{\text{qf}} \bar{c}$ that, by 2, is equivalent to $\bar{a} \equiv \bar{c}$. Let $p(\bar{x} y)$ be the set of parameter-free formulas $\varphi(\bar{x} y)$ such that $\varphi(\bar{a} b)$. Since $p(\bar{a} y)$ is finitely consistent, then $p(\bar{c} y)$ is finitely consistent. By saturation, there is a realization d of $p(\bar{c} y)$. By construction, $\bar{a} b \equiv \bar{c} d$. To show that $4 \Rightarrow 2$ suppose that that $\bar{a} \equiv_{\text{qf}} \bar{c}$. We prove by induction on the syntax of the parameter-free formula $\varphi(\bar{x})$ that $\varphi(\bar{a}) \leftrightarrow \varphi(\bar{c})$. The basic step of the induction is true by hypothesis, the case of Boolean connectives is obvious. The induction for the existential quantifier goes as follows: assume $\exists x \varphi(\bar{a} x)$ holds, then there is a b such that $\varphi(\bar{a} b)$ holds. By 4, there is a d such that $\bar{a} b \equiv_{\text{qf}} \bar{c} d$. By induction hypothesis $\varphi(\bar{c} d)$ holds. So $\exists x \varphi(\bar{c} x)$. The converse: if $\exists x \varphi(\bar{c} x)$ then $\exists x \varphi(\bar{a} x)$, holds by symmetry.

5.1.8 A test for quantifier-free definability modulo a theory.

The theorems above deal with complete theories. This seems sufficient (after all, in this course we will mainly deal with complete theories), but it is not. In fact, in many cases the only way we have to prove that a theory complete goes via showing that it eliminates quantifiers. To cover the case of incomplete theories, we need to repeat the argument in the proof of Theorem 5.1.6 in a more syntactic form.

PROPOSITION For every parameter-free formula $\varphi(\bar{x})$ the following are equivalent (recall that in this chapter *model* stands for *model of S*):

- 1 there is a parameter- and quantifier-free formula $\psi(\bar{x})$ such that $\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$ holds modulo S ; and
- 2 for every M, \bar{a} and N, \bar{c} if $M \bar{a} \equiv_{\text{qf}} N \bar{c}$ then $M \models \varphi(\bar{a})$ if and only if $N \models \varphi(\bar{c})$.
- 3 for every complete parameter- and quantifier-free type $p(\bar{x})$ either $p(\bar{x}) \rightarrow \varphi(\bar{x})$ or $p(\bar{x}) \rightarrow \neg\varphi(\bar{x})$ holds modulo S .

PROOF The implication $3 \Rightarrow 2$ is immediate: in fact $M \bar{a} \equiv_{\text{qf}} N \bar{c}$ is the same as saying that \bar{a} and \bar{c} realize the same quantifier- and parameter-free types. To prove the implication $2 \Rightarrow 3$ observe that from 2 we obtain immediately (taking $(M = N)$ a non-uniform version of 3: in every model either $p(\bar{x}) \rightarrow \varphi(\bar{x})$ or $p(\bar{x}) \rightarrow \neg\varphi(\bar{x})$. So it suffices to prove that there cannot be two models M and N such that $N \models p(\bar{x}) \rightarrow \varphi(\bar{x})$ and $N \models p(\bar{x}) \rightarrow \neg\varphi(\bar{x})$. But this also follows easily from 2. To prove the implication $3 \Rightarrow 1$ observe that from (3) it follows that in every model M

$$\varphi(\bar{M}) = \bigcup_{S \vdash p(\bar{x}) \rightarrow \varphi(\bar{x})} p(\bar{M}) \quad \text{and} \quad \neg\varphi(\bar{M}) = \bigcup_{S \vdash p(\bar{x}) \rightarrow \varphi(\bar{x})} \psi(\bar{M}).$$

where $p(\bar{x})$ ranges over the parameter- and quantifier-free types. By compactness if $p(\bar{x}) \rightarrow \varphi(\bar{x})$ holds modulo S , then already $\psi(\bar{x}) \rightarrow \varphi(\bar{x})$ holds modulo S for some $\psi(\bar{x})$ in $p(\bar{x})$. So define

$$\Psi_+ = \left\{ \psi(\bar{x}) : S \vdash \psi(\bar{x}) \rightarrow \varphi(\bar{x}) \right\} \quad \text{and} \quad \Psi_- = \left\{ \psi(\bar{x}) : S \vdash \psi(\bar{x}) \rightarrow \neg\varphi(\bar{x}) \right\}.$$

where $\psi(\bar{x})$ ranges over the parameter- and quantifier-free formulas, we have that for every model M :

$$\# \quad \varphi(\bar{M}) = \bigcup_{\psi(\bar{x}) \in \Psi_+} p(\bar{M}) \quad \text{and} \quad \neg\varphi(\bar{M}) = \bigcup_{\psi(\bar{x}) \in \Psi_-} \psi(\bar{M}).$$

Clearly for every $\psi_+(\bar{x})$ in Ψ_+ and every $\psi_-(\bar{x})$ in Ψ_- we have that $\psi_+(\bar{x}) \wedge \psi_-(\bar{x})$ is inconsistent. We claim that for some $\psi_+(\bar{x})$ and $\psi_-(\bar{x})$ we can also have $\psi_+(\bar{x}) \vee \psi_-(\bar{x})$, hence that $\psi_+(\bar{x})$ is equivalent to $\varphi(\bar{x})$. To prove the claim assume it is false and use compactness to obtain M, \bar{b} realizing the type

$$\left\{ \neg\psi(\bar{x}) : \psi(\bar{x}) \in \Psi_+ \right\} + \left\{ \neg\psi(\bar{x}) : \psi(\bar{x}) \in \Psi_- \right\}$$

But, from (#) above, \bar{b} does not belong to $\varphi(\bar{M})$ nor to $\neg\varphi(\bar{M})$, a contradiction. This proves the claim and with it the proposition.

5.1.9 Tests for quantifier elimination.

The test for elimination of quantifiers generalizes in the obvious way to incomplete theories. The proof is left to the reader.

PROPOSITION The following are equivalent (below model stands for model of S):

- 1 S has quantifier elimination;
- 2 for every M, \bar{a} and N, \bar{c} , if $M \bar{a} \text{ qf} \equiv N \bar{c}$ then $M \bar{a} \equiv N \bar{c}$;
- 3 every partial isomorphism $F : M \rightarrow N$ is an elementary map; and
- 4 for every saturated model M and N , every $\bar{a} b \subseteq M$ and $\bar{c} \subseteq N$, if $M \bar{a} \text{ qf} \equiv N \bar{c}$ then there is a $d \in N$ such that $M \bar{a} b \text{ qf} \equiv N \bar{c} d$.

5.1.10 Elimination of quantifiers and completeness.

The second equivalent in the proposition above (with for \bar{a} and \bar{c} the empty tuples) entails that any two models of S are elementary equivalent whenever they have the same characteristic. So, if S fixes the characteristic of its models (as in most examples), from elimination of quantifiers one can derive completeness.

5.1.11 An exercise.

Suppose L is finite and contains no function symbols but constants. Assume that S is complete and let M be an arbitrary model. Prove that the following are equivalent:

- 1 S has quantifier elimination;
- 2 for every $\bar{a}, \bar{c} \subseteq M$, if $\bar{a} \text{ qf} \equiv \bar{c}$ then $\bar{a} \equiv \bar{c}$; and
- 3 for every $\bar{a}, \bar{c} \subseteq M$, if $\bar{a} \text{ qf} \equiv \bar{c}$ then there is an automorphism of M that maps \bar{a} to \bar{c} .

EXERCISE

5.2 Model-completeness

It may happen that the number of alternations of quantifiers necessary to obtain all definable sets can be drastically reduced even though quantifiers cannot be completely eliminated. Model-completeness is the first degree of complexity after quantifier-free which is level zero.

5.2.1 Existential and universal formulas.

Formulas of the form $\exists \bar{y} \varphi(\bar{x} \bar{y})$ with $\varphi(\bar{x} \bar{y})$ quantifier-free are called **existential formulas** which we abbreviate by **\exists -formulas**. Formulas of the form $\forall \bar{y} \varphi(\bar{x} \bar{y})$ with $\varphi(\bar{x} \bar{y})$ quantifier-free are called **universal formula** or for short **\forall -formulas**. If T is a theory we write $\exists T$ and $\forall T$ for the set of universal, respectively, existential sentences that are consequences of T , that is the set of universal, respectively, existential sentences that hold in every structure that models T . (In particular, when T is complete this is just the set of universal, respectively, existential sentences in T .) The **universal-existential** sentences and the **existential-universal** sentences or, for short **$\forall\exists$ -formulas** and **$\exists\forall$ -formulas**, and the theories $\forall\exists T$ and $\exists\forall T$ are defined analogously.

5.2.2 One-elementarity.

When $M \subseteq N$, then $\forall \text{Th}_M M \subseteq \forall \text{Th}_M N$ is equivalent to $\exists \text{Th}_M N \subseteq \exists \text{Th}_M M$ and, since the reverse inclusions hold trivially, it is also equivalent to $\forall \text{Th}_M M = \forall \text{Th}_M N$ and $\exists \text{Th}_M M = \exists \text{Th}_M N$. If this happens we write $M \preceq_1 N$. We write $M, \bar{a} \exists \Rightarrow N, \bar{c}$ when \bar{c} satisfies in N every existential formula true in M at \bar{a} . The dual notion is $M, \bar{a} \forall \Rightarrow N, \bar{c}$ which is equivalent to $N, \bar{c} \exists \Rightarrow M, \bar{a}$. If both these equivalences obtains, i.e., if \bar{a} and \bar{b} satisfy the same existential formulas, we write $N, \bar{c} \equiv_1 M, \bar{a}$. We say that a map $F : M \rightarrow N$ is **\exists -elementary**, if $M, \bar{a} \exists \Rightarrow N, F\bar{a}$ for every \bar{a} in the support of F . The definition of **\forall -elementary** map is analogous. Clearly, $F : M \rightarrow N$ is \exists -elementary if and only if F^{-1} is \forall -elementary. Finally we say that $F : M \rightarrow N$ is **1-elementary** if it is both \exists - and \forall -elementary.

5.2.3 Model-completeness.

We say that S is **model-complete** if for every parameter-free formula $\psi(\bar{x})$ is equivalent modulo S to a parameter-free \exists -formula $\varphi(\bar{x})$ (where the tuple \bar{x} may be empty). The name model-complete may sound bizarre: it has been introduced by Abraham Robinson in the 50's, since then it has become standard. The terminology is inspired by the fact that the model-completeness of S is equivalent to the completeness of the theories $S + \text{qfTh}_M M$ for M any model (of S). This is a direct consequence of the Robinson test that we prove in 5.2.7 below.

5.2.4 A syntactic criterion.

The following is straightforward but worth of mention. As for quantifier-elimination, from this observation it follows that every model-complete theory is axiomatized by a set of universal-existential sentences.

PROPOSITION The theory S is model-complete if and only if for every parameter-free \forall -formula $\psi(\bar{x})$ is equivalent modulo S to a parameter-free \exists -formula $\varphi(\bar{x})$.

5.2.5 A test for existential definability.

This proposition is an asymmetric version of 5.1.8 above but the proof is exactly the same (substitute existential for quantifier-free and the argument will go).

PROPOSITION For every parameter-free formula $\varphi(\bar{x})$ the following are equivalent (recall that model stands for model of S):

- 1 there is a parameter-free existential formula $\psi(\bar{x})$ such that $\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$ holds modulo S ; and
- 2 for every M, \bar{a} and N, \bar{c} , if $M \bar{a} \exists \Rightarrow N \bar{c}$ and $M \models \varphi(\bar{a})$, then $N \models \varphi(\bar{c})$.

5.2.6 A forth construction.

The following theorem is proving using only one direction of a back-and-forth construction.

PROPOSITION Let M and N be models of S . Assume that N is saturated and of cardinality larger than $|M|$. Let $F : M \rightarrow N$ be a partial isomorphism. Then the following are equivalent:

- 1 F has an extension to an embedding of M into N ; and
- 2 F is \exists -elementary.

PROOF To prove the implication 1 \Rightarrow 2 let $\varphi(\bar{x})$ be a quantifier-free formula with parameters in the support of F and let $G : M \rightarrow N$ an embedding that extends F . If $\exists \bar{x} \varphi(\bar{x})$ is true in M , then $\varphi(\bar{x})$ has a solution \bar{a} in M . So $\varphi(\bar{a})$ is a quantifier-free sentence true in M . This is mapped by G to a N -formula true in N . Then $N \models G\exists \bar{x} \varphi(\bar{x})$, but this coincides with $F\exists \bar{x} \varphi(\bar{x})$. Finally, to prove 2 \Rightarrow 1 suppose that $F : M \rightarrow N$ is \exists -elementary. We use the *forth* half of a back-and-forth construction: we show that for every a there is a b in N such that $F + \langle a, b \rangle$ is also \exists -elementary. In this way we can construct a chain of maps and extend the domain of F to the whole of M . Let $p(x)$ the \exists -formulas over the support of F that are true in M at a . We claim that $Fp(x)$ is finitely consistent in N . In fact if not, then for some formula $\varphi(x)$ in $p(x)$ we would have that $N \models \neg F\exists x \varphi(x)$. But $M \models \exists x \varphi(x)$ because $\varphi(x)$ is satisfied by a , so, by hypothesis, $N \models F\exists x \varphi(x)$. So let b be a realization of $Fp(x)$ in N , and define F to be $F + \langle a, b \rangle$. We need only check that $F + \langle a, b \rangle$ is \exists -elementary. But this is clear by the choice of b .

5.2.7 Robinson test for model-completeness.

The next is a classical theorem.

PROPOSITION The following are equivalent:

- 1 S is model-complete;
- 2 for every M, \bar{a} and N, \bar{c} , if $M \bar{a} \exists \Rightarrow N \bar{c}$ then $M \bar{a} \equiv N \bar{c}$; and
- 3 $M \preceq N$ for every models such that $M \subseteq N$.

PROOF The equivalence 2 \Leftrightarrow 3 is easy: from (2) it follows that every embedding $F : M \rightarrow N$ is elementary; as a particular case we obtain (3). Vice versa, assume $M \bar{a} \exists \Rightarrow N \bar{c}$. Suppose also that N is saturated, otherwise substitute it with some saturated elementary extension. By the result above there is an embedding $F : M \rightarrow N$ such that $F\bar{a} = \bar{c}$. Then $F[M]$ is isomorphic to M , so in particular it is a model of S . So $F[M] \preceq N$ so the elementarity of $F[M] \rightarrow N$ follows immediately and with it $M \bar{a} \equiv N \bar{c}$. To prove the implication 1 \Rightarrow 3 assume $M \subseteq N$, let $\psi(\bar{x})$ be a parameter-free formula, and let $\bar{c} \subseteq M$. We want to show that if $\psi(\bar{c})$ is true in N then it is true in M . Let $\varphi(\bar{x})$ be a parameter-free formula such that

$$S \vdash \forall \bar{x} \left[\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}) \right]$$

Consequently $\varphi(\bar{c})$ holds in N so, since this is a universal formula, $\varphi(\bar{c})$ holds in M . Since the equivalence above holds also in M then $\psi(\bar{c})$ holds in M . The implication

2 \Rightarrow 1 follows immediately from 5.2.5 above.

5.2.8 Addendum.

Sometimes the Robinson test is stated in terms of 1-elementarity. Indeed simple inspection of the proof above shows the following.

PROPOSITION The following are equivalent:

- 1 S is model-complete;
- 2 for every M and N every embedding $F : M \rightarrow N$ is 1-elementary; and
- 3 $M \preceq_1 N$ whenever $M \subseteq N$.

5.2.9 Addendum.

The reader can prove as exercise that the following are equivalent:

- 1 S is model-complete; and
- 2 for every model M the theory $S +_{\text{qf}} \text{Th}_M M$ is complete over M .

EXERCISE

5.3 Examples

These are a few easy examples of how the general techniques presented in the chapter above apply to concrete cases. The proofs are merely sketched and we will assume

without proof some basic results of algebra.

5.3.1 A trivial example: infinite sets.

Let L be the empty language and let S be the theory which asserts the existence of infinitely many elements. Let M and N be two arbitrary models. Let $\bar{a} \subseteq M$ and $\bar{c} \subseteq N$ be such that $\bar{a} \equiv_{\text{qf}} \bar{c}$ (that is, \bar{a} and \bar{c} have the same equality-type) and let $b \in M$ be arbitrary. Then it is evident (simply because N is infinite) that we can find some $d \in N$ such that $\bar{c}d$ has the same equality-type of $\bar{a}b$.

5.3.2 Dense linear orders (the axioms).

Let L contain only a binary relation $<$. The theory of dense linear orders without endpoints is axiomatized by the following sentences. The axioms of linear order,

$$\begin{aligned} & \exists xy (x < y), \\ & \forall x (x \not< x), \\ & \forall xyz [(x < y \wedge y < z) \rightarrow x < z], \\ & \forall xy (x < y \vee y < x \vee x = y); \end{aligned}$$

the axiom of density,

$$\forall xy \exists z [x < y \rightarrow (x < z \wedge z < y)],$$

finally, an axiom claiming the absence of endpoints

$$\forall x \exists yz [y < x \wedge x < z].$$

5.3.3 Dense linear orders (quantifier elimination).

Let S be the theory of dense linear orders without endpoints. We claim that S has elimination of quantifiers. In fact, let M and N be two arbitrary models. Let $\bar{a} \subseteq M$ and $\bar{c} \subseteq N$ be such that $\bar{a} \equiv_{\text{qf}} \bar{c}$ and let $b \in M$ be arbitrary. If b is one of the elements of \bar{c} , it is evident that we can find $d \in N$ such that $\bar{a}b \equiv_{\text{qf}} \bar{c}d$. Otherwise, by the density and the absence of endpoints, we can find some $d \in N$ such that $\bar{c}d$ has the same order-type of $\bar{a}b$. EXERCISE

5.3.4 Random graph (the axioms).

The language contains only a binary relation E . The theory of graphs is axiomatized by the axioms

$$\begin{aligned} & \exists x (x = x), \\ & \forall x \neg E(x, x), \\ & \forall xy [E(x, y) \rightarrow E(y, x)], \end{aligned}$$

that say that E is an irreflexive symmetric relation. The theory of random graphs contains also the following group of axioms that express randomness (cf. Exercise 11.4.4 below)

$$\forall \bar{x} \forall \bar{y} \exists z \left[\bigwedge_{i < n, j < m} x_i \neq y_j \rightarrow \bigwedge_{i < n, j < m} E(x_i, z) \wedge \neg E(y_j, z) \right]$$

for every tuple of variables $\bar{x} = x_0 \cdots x_{n-1}$ and $\bar{y} = y_0 \cdots y_{m-1}$ of positive arity. Observe that every countable model of these axioms is a random graph.

5.3.5 Random graph (quantifier elimination).

The reader can prove as exercise that the theory of random graphs has elimination of quantifiers. (The proof is similar to the proof for dense linear orders; here randomness plays the role of density.) The reader can also show that every saturated model of the axioms above is a random graph (but there are uncountable models of these axioms that are not random graphs, that is, they are not generic structures in the sense of Chapter 11 above). EXERCISE

5.3.6 Vector spaces (the axioms).

Let K be a field. Let L_K be the language of the vector spaces over K . The operations of addition and multiplication in K are denoted by $+_K$ and \cdot_K , these lives in the metalanguage and should not be confused with the symbols of L_K . The theory of vector spaces contains the axioms of Abelian groups:

$$\begin{aligned} & \forall xyz [(x + y) + z = x + (y + z)], \\ & \forall x [(0 + x = x) \wedge (x + 0 = x)], \\ & \forall x [x - x = 0], \end{aligned}$$

and, for every $k, h \in K$:

$$\begin{aligned} & \forall xy [k(x + y) = (kx) + (ky)], \\ & \forall xy [k(-x) = (-_K k), x], \\ & \forall x [(kx) + (hx) = (k +_K h), x], \\ & \forall x [k(hx) = (k \cdot_K h) x], \end{aligned}$$

The notation may be confusing: the expressions $-_K k$, $k +_K h$ and $k \cdot_K h$ stand for elements of K and therefore for function symbols of L_K , they are not terms as, for instance, $k(-x)$ and $kx + hx$.

Let M be a structure satisfying the axioms above. Let F be the set of functions that interpret the elements of K . We can define sum $+_F$ and multiplication \cdot_F on F : we let $k +_F h$ be the function that maps x to $(kx) + (hx)$ and let $k \cdot_F h$ be the

function that maps x to $k(hx)$. The reader can check that with these operations make F a field isomorphic to K , and that M is a vector space over F .

5.3.7 Vector spaces (elimination of quantifier).

Let S be the theory of vector spaces over an infinite field K (we could allow finite fields but only if we add to the axioms above some infinitely many sentences that exclude finite models). Up to equivalence, atomic formulas have the form $t(\bar{x}) = 0$ where $t(\bar{x})$ is linear combination of the vectors \bar{x} . Let M and N be saturated. Let \bar{a} and \bar{c} be arbitrary tuples of M , respectively, N such that $M, \bar{a} \equiv N, \bar{c}$, we want to prove that for every $b \in M$ there is a $d \in N$ such that $M, \bar{a} b \equiv N, \bar{c} d$. We distinguish two cases: first suppose that $t(\bar{a}) = b$ for some parameter-free term $t(\bar{x})$. Then it is immediate that $d := t(\bar{c})$ will do. Second suppose $t(\bar{a}) \neq b$ for all parameter-free term $t(\bar{x})$. Then we choose $d \in N$ such that $t(\bar{c}) \neq d$ for all parameter-free term $t(\bar{x})$. This d exists by saturation. To check that $M, \bar{a} b \equiv N, \bar{c} d$ we need to check that for every parameter-free term $t(\bar{x})$ and for every $k \in K$ we have $t(\bar{a}) + kb = 0$ iff $t(\bar{c}) + kd = 0$. As b and d are linearly independent from \bar{a} , respectively \bar{c} , this follows immediately from the induction hypothesis.

5.3.8 Applications

The following are two very easy consequences of quantifier elimination. Note that without using elimination of quantifiers, these claims would be heavier to prove. Fix an infinite field K , then:

- 1 the theory of vector spaces (over K) is complete; and
- 2 if N is a vector space and M is a linear subspace of N then $M \preceq N$.

5.3.9 Atomless Boolean algebra of sets.

Let L contain for every $i \in \omega$ an unary relation symbol r_i . Let T be the theory in 4.3.8 above. As an exercise one can prove that T has quantifier elimination. Use this result to prove that T is complete. EXERCISE

5.3.10 An exercise.

Let L contain no function symbols of positive arity and suppose that L contains relation symbols of arity at most 1. Then every complete theory has elimination of quantifiers. EXERCISE

5.3.11 An exercise.

Let L contain a unary relation symbols r and a binary relation symbol $<$. Let T contain the sentences asserting that $<$ is a linear order without endpoints and that

whenever $a < b$ there are c and d such that $a < c < d < b$ such that $r(c)$ and $\neg r(d)$. Prove that T has elimination of quantifiers. EXERCISE

5.3.12 An exercise.

Let N be structure with domain the natural numbers, with one unary function: the successor, and one binary relation: the order. Prove that $\text{Th}N$ has quantifier elimination. Give an explicit axiomatization of $\text{Th}N$. EXERCISE

5.3.13 Algebraically closed fields (the axioms).

Let L be the language of rings. The theory of **fields** is axiomatized by the following group of axioms

$$\begin{aligned} & \forall xyz [(xy)z = x(yz)], \\ & \forall x [x1 = x \wedge 1x = x], \\ & \forall xy [xy = yx], \\ & \forall xyz [(x + y)z = xz + yz], \\ & \forall x \exists y [x \neq 0 \rightarrow xy = 1]; \end{aligned}$$

These sentences do not axiomatize a complete theory, in fact, they do not even decide all parameter- and quantifier-free sentences. To have that we either have to add, for some $p > 0$, the axiom

$$1 + \dots (p \text{ times}) \dots + 1 = 1.$$

or for every positive p the axioms

$$1 + \dots (p \text{ times}) \dots + 1 \neq 0.$$

in the first case we obtain the theory of fields of **characteristic p** , in the second case the field is said to have **characteristic 0**. The theory of **algebraically closed fields of characteristic p** . This is axiomatized by the axioms above and the sentences

$$\forall x_1 \dots x_n \exists y [y^n + x_1 y^{n-1} + \dots + x_n = 0].$$

for every positive n . Finally observe that in every field of characteristic p we can embed the prime field of characteristic p . All structures that model the theory of fields of characteristic p have the same characteristic in the sense of Definition 1.1.10 above. So the result below implies that the theory of algebraically closed fields of a fixed characteristic is complete.

5.3.14 Algebraically closed fields (elimination of quantifiers).

Logicians credit the proof of quantifier elimination in algebraically closed fields to Alfred Tarski. The theorem is also well-known to algebraists, only, they do not

call it quantifier elimination and credit it to Claude Chevalley. Here we sketch a non-constructive (but simple) proof of quantifier elimination for algebraically closed field.

Fix a characteristic p and let S be the theory of algebraically closed fields of characteristic p . We need the following fact from algebra. Let K be a model (so, a field of characteristic p). Let $K[x]$ denote the set of K -terms $t(x)$ where no variable but x occurs (that is, the ring of polynomials in x over K). Let I be a non-empty set of $K[x]$ that is closed under $+$ and under multiplication by elements of $K[x]$ (that is, an non-trivial ideal). Then there is $t(x) \in K[x]$ such that every element of I has the form $s(x) \cdot t(x)$ for some $s(x) \in K[x]$ (that is, I is principal). For the proof one may consult any text of undergraduate algebra.

Now, if a is algebraic over K then the set of polynomial $p(x)$ such that $p(a) = 0$ is an ideal I of $K[x]$. So I is generated by some term $t(x)$ is as above. The polynomial $t(x)$ has the property that if $t(x) = p(x) \cdot q(x)$ then either $p(x)$ or $q(x)$ is a constant polynomial. In fact, if $t(x) = p(x) \cdot q(x)$ then at least one of $p(x)$ and $q(x)$ vanish at a . Suppose $p(a) = 0$, so $p(x)$ belongs to I and it factorize to $s(x) \cdot t(x)$ for some $s(x)$. Then $t(x) = q(x) \cdot s(x) \cdot t(x)$ is only possible if $s(x)$ and $q(x)$ are constant polynomials.

If a and $t(x)$ are like above, and b is such that $t(b) = 0$, then for every polynomial $p(x)$ we have: $p(b) = 0$ if and only if $p(a) = 0$. In fact, if $p(a) = 0$ then $p(x) = s(x) \cdot t(x)$ for some $s(x)$ and, since $t(b) = 0$ then also $p(b) = 0$. Vice versa, if J is the ideal of polynomial that vanish at b and $r(x)$ is a generator of J . Then $t(x) = r(x) \cdot s(x)$ for some $s(x)$. So $r(x)$ and $t(x)$ are equal up to a constant, so $I = J$.

From this it follows immediately that $a \equiv_K b$ holds if and only if $t(b) = 0$. In fact, atomic formulas have the form $t_1(x) = t_2(x)$ but, this is equivalent to the formula $t_1(x) - t_2(x) = 0$, so we can without loss of generality consider only atomic formulas of the form $t(x) = 0$.) The same argument also shows that if $F : M \rightarrow N$ is a partial isomorphism and $\text{dom } F = K$, then $M, a \equiv N, b$ if and only if b is a zero of $Ft(x)$.

This been said, we proceed in the proof of quantifier elimination. Let \bar{a} and \bar{c} be arbitrary tuples of M , respectively, N such that $M, \bar{a} \equiv N, \bar{c}$, we want to prove that for every $b \in M$ there is a $d \in N$ such that $M, \bar{a} b \equiv N, \bar{c} d$. Let $K \subseteq M$ be the field generated by \bar{a} (in algebra this is called the field of fractions and is denoted by $K(\bar{a})$). It is clear that the partial isomorphism $\bar{a} \mapsto \bar{c}$ extends in a unique way to an isomorphism F with domain K . Now, there are two cases: b is the zero of some polynomial with coefficients K . Let $t(x)$ be this polynomial, by the remark above we can assume that $t(x) = 0$ decides every atomic K -formula with free variable x . Since K and $F[K]$ are isomorphic structure the same holds for the formula $F(t(x) = 0)$. Since N is algebraically closed, $t(x) = 0$ has a solution, d .

So $M, \bar{a} b \equiv N, \bar{c} d$. The second case is that b is transcendental (i.e. non-algebraic) over K . Polynomials have only finitely many solutions, there are countably many polynomials over \bar{c} so, since N is uncountable, there is a d which is transcendental over \bar{b} . We may conclude that $M, \bar{a} b \equiv N, \bar{c} d$.

5.3.15 Real closed fields (the axioms).

The language of ordered fields is the language of fields expanded with the binary relation $<$. The theory of ordered fields is the theory of fields plus the following sets of axioms. The axioms of linear orders (see 5.3.2 above) plus the axioms of fields (see 5.3.13 above), and the following two sentences:

$$\begin{aligned} \forall xy [(0 < x \wedge 0 < y) \rightarrow 0 < xy], \\ \forall xyz (x < y \rightarrow x + z < y + z), \end{aligned}$$

axiomatizes the theory of **ordered fields**. The theory of **real closed fields** contains also the axioms:

$$\begin{aligned} \forall x \exists y (0 < x \rightarrow x = y^2), \\ \forall x_1 \cdots x_n \exists y [y^n + x_1 y^{n-1} + \cdots + x_n = 0] \text{ for every odd integer } n > 0. \end{aligned}$$

and an infinite set of sentences like in 5.3.13 above that say that the field has characteristic 0.

5.3.16 Real closed fields (elimination of quantifier).

We prove that the theory of real closed field has elimination of quantifiers. Atomic formulas have the form $t = s$ or $t < s$ for some term s and t . Clearly these are equivalent to formulas of the form $t - s = 0$ and $t - s < 0$. So $\bar{a} \equiv \bar{c}$ if and only if $t(\bar{a})$ and $t(\bar{c})$ have the same sign (positive, negative, or zero) for every term $t(\bar{x})$. In the language of real closed field parameters-free terms are just polynomial over the integers. But observe that every model contains (a copy of) the rationals Q and that $\bar{a} \equiv \bar{c}$ if and only if $\bar{a} \equiv_Q \bar{c}$, that is, $t(\bar{a})$ and $t(\bar{c})$ have the same sign for every Q -term $t(\bar{x})$. Clearly Q -terms are just polynomials over the rationals. So fix two saturated models M and N and let $\bar{a}, \bar{b} \subseteq M$ and $\bar{c} \subseteq N$ be such that $M, \bar{a} \equiv N, \bar{c}$. To prove that for some d we have that $M, \bar{a} b \equiv_Q N, \bar{c} d$. Otherwise, observe that every formula of the form $t(\bar{x} z) < 0$ and $t(\bar{x} z) = 0$ is equivalent to a formula of the forms $s(\bar{x}) < z$, respectively, $s(\bar{x}) = z$ for some Q -term $s(\bar{x})$. So we distinguish two cases. First, suppose that $s(\bar{a}) = b$ for some term $s(\bar{x})$. Then it is immediate that $d := s(\bar{c})$ will do. Let $p(\bar{a} z)$ be the set of the formulas $t_1(\bar{a}) < z < t_2(\bar{a})$, for $t_1(\bar{x})$ and $t_2(\bar{x})$ some Q -terms, that are satisfied by b . By the induction hypothesis $t_1(\bar{c}) < t_2(\bar{c})$ so by the density of the order, we deduce that $p(\bar{c} z)$ is finitely consistent. Let d be any realization of $p(\bar{c} z)$. It follows that $\bar{a} b \equiv_Q \bar{c} d$.

Chapter 6

Saturation and homogeneity

In the first section we redo in part what we did in Section 4.3 from a different perspective. In this chapter models are all elementary substructures of one large monster model. So each model can be safely identified with its domain and any set of parameters of small cardinality is contained in some model. This lightens the notation. In the second section the notion of homogeneous structure.

Notation and global assumptions.

Fix a signature L and a complete theory T . Fix also a monster model \mathcal{U} (a.k.a. universal domain) of T . That is, \mathcal{U} is saturated model of cardinality κ , where κ is a large cardinal larger than $|T|$ (that is, $|LN_0|$). We shall assume the existence of κ inaccessible cardinals below κ (this is not really necessary but it smoothes the exposition). Cardinals that are less than κ are called **small**. By **model** we mean an elementary substructure of \mathcal{U} of small cardinality. The symbols M, N , etc. are reserved for models. We use the symbols A, B , etc. to denote sets of parameters, that is, subsets of \mathcal{U} , of small cardinality. Sets of larger cardinality are denoted either by calligraphic letters like \mathcal{D}, \mathcal{C} etc., or by writing $\varphi(\bar{U})$ where $\varphi(\bar{x})$ is the defining formula. We write $\bar{a} \equiv_A \bar{c}$ if the tuple \bar{a} and \bar{c} have the same type over A . We denote definable sets by writing $\varphi(\bar{U})$ where $\varphi(\bar{x})$ is the defining formula. A similar notation is used for type-definable sets: we write $p(\bar{U})$ for the set defined by $p(\bar{x})$. The **truth** of sentences and the **consistency** of formulas is always evaluated in \mathcal{U} . By elementarity we can think these are evaluated in any model as soon as this model contains all the parameters. We write $\varphi(\bar{M})$ for $\varphi(\bar{U}) \cap \bar{M}$: the set of solutions of $\varphi(\bar{x})$ that belong to M . Sets of the form $\varphi(\bar{M})$, where $\varphi(\bar{x})$ is over M , are called **definable subsets** of M . When the parameters of $\varphi(\bar{x})$ belong to some set A we say that $\varphi(\bar{M})$ is **definable over A** .

6.1 Saturation

We have already met the notion of saturation in the previous chapters. In this section we examine it more systematically. First we recall the definition of types and realizations. Observe that here we add an extra claim on the cardinality of the types and require consistency to be evaluated in the universe.

6.1.1 Types.

A **type** is a consistent set of formulas $p(\bar{x})$ of small cardinality with free variables among some finite tuple \bar{x} . (Types have been defined also in 4.1.2 above. There we only required consistency. When working in the monster model, we reserve the word *type* to set of formulas that are consistent *with T* , or equivalently, consistent *in \mathcal{U}* .) If all formulas in $p(\bar{x})$ are over A , then we may say **type over A** or, for short, **A -type**. We always assume types are closed under conjunctions though we will not be pedantic about that: when convenient, we confuse a consistent set of formulas with its closure under conjunction. Observe that, by the saturation of the universe, the requirement of consistency can be replaced with finite consistency. An A -type is **complete** if for every A -formula $\varphi(\bar{x})$ either $\varphi(\bar{x})$ or $\neg\varphi(\bar{x})$ is in the type. (Some authors include completeness as a requirement in the definition of type. Then they say **partial** type for what here we call a type.) The set of complete A -types is denoted by $\mathbf{S}(A)$ or, if we restrict to types with n free variables, by $\mathbf{S}_n(A)$.

6.1.2 Realizations.

We say that \bar{a} **realizes (or, satisfies) the type $p(\bar{x})$** if $\varphi(\bar{a})$ holds for every $\varphi \in p$. We denote by $p(\bar{U})$ the set of realizations of $p(\bar{x})$, that is,

$$p(\bar{U}) = \bigcap_{\varphi \in p} \varphi(\bar{U}).$$

Sometimes we may identify a type with the set of its realizations. We say that M **realizes $p(\bar{x})$** if some realization of $p(\bar{x})$ belongs to M . When the parameters of A are in M then, by elementarity, this coincides with the notion of realization defined in 4.1.1 above.

6.1.3 The type of a tuple.

Let \bar{a} be a tuple with the same arity as \bar{x} . The set of A -formulas $\varphi(\bar{x})$ such that $\varphi(\bar{a})$ holds, is an A -type. This is called **the type of \bar{a} over A** or **the A -type of \bar{a}** . A notation frequently used in the literature for the A -type of \bar{a} is **$\text{tp}(\bar{a}/A)$** .

6.1.4 Elementary maps.

If $F : M \rightarrow N$ is an elementary map (between models), then $F : \mathcal{U} \rightarrow \mathcal{U}$ is an elementary map, and also $F : M' \rightarrow N'$ for every model containing the domain of definition and the range of F . So we domain and codomain are not very relevant we simply say: F is an elementary map.

Recall (see 3.2.6 above) that to an elementary map we associate a mapping of formulas (or set of formulas) over $\text{dom } F$ to formulas (respectively, sets of formulas) over $\text{rng } F$. This mapping preserves the consistency of formulas and, consequently, the finite consistency of set of formulas. By saturation, consistency is equivalent to finite consistency so, if $p(\bar{x})$ is a type over $\text{dom } F$, then $Fp(\bar{x})$ is a type over $\text{rng } F$.

6.1.5 Saturation.

We say that a model M is λ -saturated if for every set $C \subseteq M$ of cardinality $< \lambda$ every C -type is realized in M . The most interesting value of λ is the cardinality of M itself so this is taken as default value: **M is saturated** means that M is $|M|$ -saturated. (This is the same notion as in 4.3.1 above)

6.1.6 Existence of saturated models.

The existence of λ -saturated models is problematic: it may depend on the combinatorial properties of the cardinal λ . The situation for saturated models is even worse: their existence depends on set theoretical assumptions such as the generalized continuum hypothesis or the existence of uncountable inaccessible cardinals. A fairly general theorem is stated and proved in [HODGES]. Life is simplified if we assume λ to be inaccessible, then the existence of saturated model. To obtain a saturated model (recall that in this chapter a *model* is an elementary substructure of \mathcal{U} of small cardinality) it suffices to repeat the construction of the downward Löwenheim-Skolem theorem but using types instead of formulas.

PROPOSITION Let λ be an inaccessible cardinal larger than $|T|$ and let A be a set of cardinality $\leq \lambda$. Then there exist a saturated model M of cardinality λ containing A .

PROOF We let M be the limit of a chain of models of length λ . All models in the chain have cardinality less than λ . The chain starts with M and at each limit stages α we take the union of the previous stages and by the regularity of λ we preserve the induction hypothesis: $|M_\alpha| < \lambda$. At stage $\alpha + 1$ we let $M_{\alpha+1}$ be any model that realizes every complete M_α -type. Since there are at most $2^{|T M_\alpha|}$ different complete types over M_α , by the inaccessibility of λ , we can also require that $|M_{\alpha+1}| < \lambda$. Now, to see that the limit M is saturated, fix a C -type $p(\bar{x})$ for some $C \subseteq M$ of cardinality less than λ . By the regularity of λ , there is a $\alpha < \lambda$ such that C is contained in M_α . So $p(\bar{x})$ is realized in $M_{\alpha+1}$ and hence in M .

6.1.7 Countable saturated models.

Saturated models need not necessarily have large cardinality. This depends on the theory T . The following proposition is useful in relation to prime and atomic models (cf. Chapter 10 below). We say that the theory T is **small** if there are at most countably many parameter-free complete types.

PROPOSITION The following are equivalent:

- 1 there is a countable saturated model; and
- 2 T is small.

PROOF We sketch a direct proof but note that one could alternatively use Proposition 6.2.13 and 6.2.18 below. First observe that there are countably many complete types over any finite tuple of parameters \bar{a} . In fact a complete type over \bar{a} can be written as $p(\bar{a} \bar{x})$ where $p(\bar{z} \bar{x})$ is a complete parameters-free type. To construct a countable saturated model one proceeds like in the proof of the downward Löwenheim-Skolem theorem but with realizations of types over finitely many parameters in place of solutions of formulas.

6.1.8 Uniqueness of saturated models.

The following fact is just a corollary of the classification of homogeneous models that we obtain in 6.2.16 below. But a direct proof is also easy; it motivates the concept of homogeneous model that we introduce in the next section.

PROPOSITION Any two saturated models of the same cardinality are isomorphic.

PROOF Let M and N be two saturated models of the same cardinality λ . We define with a back-and-forth construction a chain of elementary maps $F_\alpha : M \rightarrow N$ that gives in the limit an isomorphism. The chain starts with the empty map and is the union at limit stages. For α even, we let a_α be the least element of $M \setminus \text{dom } F_\alpha$ in some arbitrary but fix well-ordering of M . Let $p(x)$ be the type of a over $\text{dom } F_\alpha$ and let b be a realization in N of $Fp(x)$. Saturation guarantees the existence of b . Finally, let $F_{\alpha+1}$ be the map $F_\alpha + \langle a, b \rangle$. The reader can easily verify that this is an elementary map. At odd stages we act similarly but exchange M with N and substitute F_α^{-1} for F_α .

6.2 Homogeneity

Saturation implies the existence of many symmetries: every tuple \bar{a} can be mapped, via an automorphism, to any other tuple that is not distinguishable from \bar{a} by a first-order formula. This property is called elementary homogeneity (precisely, elementary ω -homogeneity).

6.2.1 Homogeneous models.

Let λ be an infinite cardinal. We say that a model M is **elementarily λ -homogeneous** if every elementary map $F : M \rightarrow M$ of cardinality less than λ extends to a **automorphism of M** . The cardinals ω and $|M|$ are for us the most interesting to substitute for λ . Rephrasing the definition above in the case when λ is ω we obtain that M is ω -elementarily homogeneous if for every $\bar{a}, \bar{b} \subseteq M$ such that $\bar{a} \equiv \bar{b}$ (i.e. with the same parameter-free type) there is an automorphisms of M that maps \bar{a} to \bar{b} . As for the notion of saturation, we take $|M|$ as the default value of λ : we say **elementarily homogeneous** for $|M|$ -elementarily homogeneous. It is also usual to omit to specify *elementary* and just say *homogeneous*. But it should be emphasized that the requirement of elementarity, which occurs in all definitions above, cannot be omitted. Indeed if one replaces *elementary map* with *partial isomorphism* in the definition above than one obtains quantifier-free homogeneity: a different notion. Every theory has an elementarily homogeneous model (cf. Proposition 6.2.12 below). On the contrary quantifier-free homogeneous models may exist or not depending on the theory.

6.2.2 Examples and non-examples.

Let T be the theory of the dense linear orders without endpoints. By Cantor's back-and-forth argument, countable models are homogeneous. The model that contains a copy of the reals followed by a copy of the rationals is *not* homogeneous: all elements have the same type over \emptyset but no automorphism can map a real to a rational.

6.2.3 Back-and-forth extendible maps.

The terminology in this paragraph is non-standard. Let $F : M \rightarrow N$ be an elementary map. We say that $F : M \rightarrow N$ is **(finitely) extendible** if for every $\bar{a} \subseteq M$ there is a $\bar{b} \subseteq N$ such that $F + \langle \bar{a}, \bar{b} \rangle$ is elementary. We have denoted by $\langle \bar{a}, \bar{b} \rangle$ the tuple of pairs $\langle a_1, b_1 \rangle \cdots \langle a_n, b_n \rangle$ obtained by pairing the components of the tuples $\bar{a} = a_1 \cdots a_n$ and $\bar{b} = b_1 \cdots b_n$. We say that $F : M \rightarrow N$ is **back-and-forth** extendible if both $F : M \rightarrow N$ and $F^{-1} : N \rightarrow M$ are extendible.

6.2.4 Extendible \leftrightarrow preserving consistency of types.

Defining extendible maps, we had in mind concrete applications: back-and-forth constructions. We opted for a plug-'n-play definition. Now we show that extendible maps are not an ad hoc notion but a very natural generalization of elementary maps. Recall that elementary maps are maps that preserve consistency of formulas, in particular, they preserve the property of being a type. Extendible elementary maps preserve also the realizability of types.

PROPOSITION Let $F : M \rightarrow N$ be an elementary map. The following are equivalent:

- 1 $F : M \rightarrow N$ is extendible; and
- 2 $F : M \rightarrow N$ maps types over $\text{dom } F$ that are realized in M to types over $\text{rng } F$ that are realized in N .

PROOF To prove 1 \Rightarrow 2 let $F : M \rightarrow N$ be extendible. Let $p(\bar{x})$ be a type over $\text{dom } F$ which is realized in M . Let $\bar{a} \subseteq M$ realize $p(\bar{x})$ and let $\bar{b} \in N$ be such that $F + \langle \bar{a}, \bar{b} \rangle$ is an elementary map. Clearly \bar{b} realizes $Fp(\bar{x})$. To prove 2 \Rightarrow 1 assume 2. Let $\bar{a} \subseteq M$ be arbitrary and let $p(\bar{x})$ be the type of \bar{a} over $\text{dom } F$. Choose some $\bar{b} \subseteq N$ that realizes $Fp(\bar{x})$. It is easy to check that $F + \langle \bar{a}, \bar{b} \rangle$ is an elementary map, in fact, if $\varphi(\bar{a})$ is a true $\text{dom } F$ -sentence, then $\varphi(\bar{x})$ belongs to $p(\bar{x})$ and, consequently, \bar{b} satisfies $F\varphi(\bar{x})$.

6.2.5 A non-example.

There are elementary maps that are not extendible. Let T be the theory of dense linear orders without end-points. Let M be a model isomorphic to the real line and let $F : M \rightarrow M$ stretch the (copy in M of) the rational interval $(0, 1)$ to the whole rational line. By elimination of quantifiers, this is an elementary map but clearly it is not extendible.

6.2.6 Homogeneity over some parameters.

Let A be an arbitrary set of parameters. For models containing A we can define the notion of A -homogeneity simply replacing elementarity with A -elementarity in the definitions above. To lighten the notation we discuss only the parameter-free version of homogeneity but observe that all definitions and results in this chapter easily generalize. To include parameters it suffices to apply the following two systematic changes: 1 for all models under consideration we require that they contain A ; and 2 in all hypothesis $|T|$ is replaced with $|TA|$. This generalization is clear: parameters can always be absorbed in the language and considered as constants.

6.2.7 Homogeneous \leftrightarrow elementary maps are extendible.

To check that a model is homogeneous one frequently construct automorphisms by a back-and-forth construction. The following fact isolates the common idea of these arguments.

PROPOSITION For every model M the following are equivalent:

- 1 M is homogeneous; and
- 2 every elementary map $F : M \rightarrow M$ of cardinality $< |M|$ is extendible.

PROOF The implication 1 \Rightarrow 2 is clear: if H is an automorphism of M which extends

F , then for every $\bar{a}, \bar{b} \subseteq M$ the functions $F + \langle \bar{a}, H\bar{a} \rangle$ is an elementary map (since $F \subseteq H$, this is a well-defined function). The implication [2 \$\Rightarrow\$ 1](#) is proved using the so-called back-and-forth method. Let $F : M \rightarrow M$ be an elementary map of cardinality $< |M|$, the automorphism that extends F is constructed as limit of a chain of elementary maps. The chain has length $|M|$, each map in the chain has cardinality $< |M|$. The first map of the chain is F . At stage α even, let a be the least element of M (w.r.t. some fixed a well-ordering of M) not in $\text{dom } F_\alpha$, and let $F_{\alpha+1}$ be $F_\alpha + \langle a, b \rangle$ where $b \in M$ is such that $F_{\alpha+1}$ is an elementary map. For α odd, we let b be the least element of M not in $\text{rng } F_\alpha$. Using that $F_\alpha^{-1} : M \rightarrow M$ is also elementary (hence is extendible), we find in a similar way the element $a \in M$ to be added to $\text{dom } F_\alpha$. At limit stages, we take the union.

6.2.8 Saturated \rightarrow homogeneous.

Now we show that saturated models are homogeneous. The converse implication is not true, in Exercises [6.2.18](#) below we see that only a small amount of saturation is required for a homogeneous model to be saturated.

PROPOSITION If M is saturated then M is homogeneous.

PROOF By the characterization of homogeneity given in Proposition [6.2.7](#), it suffices to show that every elementary map $F : M \rightarrow M$ of cardinality less than $|M|$ is extendible in M . Let $F : M \rightarrow M$ be an elementary map such that $|F| < |M|$. So by Proposition [6.2.4](#) it suffices to show that F preserves realizability in M of types over $\text{dom } F$. Let $p(\bar{x})$ be a type over $\text{dom } F$. If we can show that $Fp(\bar{x})$ is also a type (i.e. it is consistent) then saturation guarantees that $Fp(\bar{x})$ is realized in M . Let $\varphi(\bar{x})$ be formula in $p(\bar{x})$. Then $\exists \bar{x} \varphi(\bar{x})$ is true (we are using that all parameters of $\varphi(\bar{x})$ are in M). Since F is elementary, $F(\exists \bar{x} \varphi(\bar{x}))$ is also true. This proves that $Fp(\bar{x})$ is a type.

6.2.9 A counter-example.

It is easy to see that the implication in the proposition above cannot be reversed. Let L contain a binary relation $<$ and a unary predicates r_i for each $i \in \omega$. Let M be the rational numbers with the usual order relation and, as interpretation of the relations r_i , the open intervals (i, ∞) . It is not difficult to see that M is homogeneous. Clearly M is not saturated.

6.2.10 The universe is homogeneous.

In the proof above we do not use that M is of small cardinality. So the proposition above applies also to the universe. This is an important fact that will be used in the sequel without explicit mention.

PROPOSITION The universe \mathcal{U} is elementarily homogeneous.

6.2.11 Types and orbits.

As a consequence of the proposition above we have a useful representation of complete types in terms of orbits under automorphisms. Let $\text{Aut}_A \mathcal{U}$ denote the set of A -automorphisms of the universe, that is, those automorphisms that fix A pointwise. The **A -orbit of the type \bar{a}** is the set

$$O_A(\bar{a}) := \{F\bar{a} : F \in \text{Aut}_A \mathcal{U}\}$$

The reader can prove as an exercise that the orbit $O_A(\bar{a})$ equals $p(\bar{\mathcal{U}})$ where $p(\bar{x})$ is the A -type of \bar{a} . **EXERCISE**

6.2.12 Existence of homogeneous models.

By Proposition [6.2.8](#) above, we can derive the existence of homogeneous models from the existence of saturated models. But one need not go through saturation to get homogeneity. Homogeneous structures need not be as large as saturated ones. Indeed observe that in the following theorem we do not require λ to be *strictly* larger than $|T|$. This allows the existence of countable homogeneous models (this is the case we are most interested in). As for the proof of the existence of saturated models. To this theorem the same remark as in [6.1.6](#) apply: we do not prove the most general theorem but refer to [[HODGES](#)] for a more detailed version.

PROPOSITION Let A be a set and let λ be an inaccessible cardinal such that $|TA| \leq \lambda$. Then A is contained in an homogeneous model M of cardinality λ .

PROOF We can derive the existence of M for the existence of saturated models and the fact that saturated models are homogeneous. But note that Proposition [6.2.8](#) assumes $|T| < \lambda$. So the case $|T| = \lambda$ has to be proved. When T is countable this case is covered by Proposition [6.2.13](#) below. The case when $|T|$ is an uncountable inaccessible cardinal (a case we are not primarily interested in) can be entrusted to the reader (who can use the proof in [6.2.13](#) for inspiration).

6.2.13 Existence of countably homogeneous models.

In many proof the use of countable homogeneity may be used in place of homogeneity. An advantage is that ω -homogeneous models need not be too large.

PROPOSITION Let A be a set. Then A is contained in an ω -homogeneous model M of cardinality $|TA|$.

PROOF We let M be the union a chain $\langle A_i : i \in \omega \rangle$ of sets of cardinality $|TA|$. The chain starts with A . At stage i for i even we work to obtain in the limit a model so, as in the proof of the Löwenheim-Skolem theorem, we take action to satisfy the Tarski-Vaught test: let A_{i+1} contain a solution for every consistent A_i -formula.

When i is odd, we work for ω -homogeneity. Consider all pairs $\langle F, \bar{a} \rangle$ where $\bar{a} \subseteq A_i$ and F is a finite elementary map with domain of definition and range contained in A_i . For each such pair let A_{i+1} contain a realization of $Fp(\bar{x})$, where $p(\bar{x})$ be the type of \bar{a} over $\text{dom } F$. Finally we show that M is ω -homogeneous. Fix some arbitrary finite elementary map $F : M \rightarrow M$ and some $\bar{a} \subseteq M$. Clearly, there is an α such that $\bar{a} + \text{dom } F \subseteq A_i$. Let $p(\bar{x})$ be the type of \bar{a} over $\text{dom } F$. From the construction it is clear that some realization of $Fp(\bar{x})$ will enter M at the next odd stage. So $F + \langle \bar{a}, \bar{b} \rangle$ is also an elementary map by the characterization in Proposition 6.2.7 above.

6.2.14 Extendible maps between homogeneous models.

The proof of proposition 6.2.16 below requires two lemmas (that will take care of the successor and, respectively, the limit step of the induction).

PROPOSITION Let N be ω -homogeneous and let $F : M \rightarrow N$ be an extendible elementary map of cardinality $< |N|$. Then every elementary map $H : M \rightarrow N$ which is a finite extension of $F : M \rightarrow N$ is also extendible.

PROOF Suppose F is extendible and let $F + \langle \bar{a}, \bar{b} \rangle$ be an elementary map. We claim that $F + \langle \bar{a}, \bar{b} \rangle$ is extendible. Let $\bar{a}' \subseteq M$ be arbitrary. Since F is extendible, there is a tuple $\bar{d}\bar{d}' \subseteq N$ such that $F + \langle \bar{a}\bar{a}', \bar{d}\bar{d}' \rangle$ is an elementary map. The tuple \bar{d} has the same type over $\text{rng } F$ as \bar{b} so, by homogeneity, there is an automorphism H of N that fixes $\text{rng } F$ and maps \bar{d} to \bar{b} . Let $\bar{b}' = H\bar{d}'$. Then $F + \langle \bar{a}\bar{a}', \bar{b}\bar{b}' \rangle$ is an elementary map.

6.2.15 A chain lemma for extendible maps between homogeneous models.

The limit of a chain of extendible maps need not be extendible, for example, consider the non-extendible elementary map constructed in 6.2.5 above: this is easily obtained as limit of a countable chain of extendible maps (take finite maps). The next lemma is trickier than the chain lemmas we have met so far. The trick is similar to the one used above (based on approximations and corrections).

PROPOSITION Let N be homogeneous and let $F_\alpha : M \rightarrow N$ be a chain of extendible elementary maps. Suppose that F , the limit of the chain, has cardinality $< |N|$. Then F is extendible.

PROOF We shall prove the proposition for chains of the form $F_{\alpha+1} = F_\alpha + \langle a_\alpha, b_\alpha \rangle$ for some pair $\langle a_\alpha, b_\alpha \rangle$ (i.e. chains that grow one-by-one) and leave to the reader to show that this is general enough. By the chain lemma for elementary maps, F is elementary, so we only need to prove extendibility. We argue by induction on the length of the chain λ (an ordinal); by the proposition above we need only consider the limit case. Given $\bar{a} \subseteq M$ we want to find a $\bar{b} \subseteq N$ such that $F + \langle \bar{a}, \bar{b} \rangle$ is

elementary. Define inductively the chain H_α of length λ . Let H_0 be $F_0 + \langle \bar{a}, \bar{b}^* \rangle$ where $\bar{b}^* \subseteq N$ is such that $F_0 + \langle \bar{a}, \bar{b}^* \rangle$ is elementary. For every $\alpha < \lambda$ define $H_{\alpha+1} = H_\alpha + \langle a_\alpha, d_\alpha \rangle$, where d_α is chosen so that $H_{\alpha+1}$ is elementary. This we can find since by Proposition 6.2.14 and the induction hypothesis, H_α is a chain of extendible maps. The union H of the chain is an elementary map: it extends F_0 but does not extend F . Now we use the homogeneity of N to correct H . Observe that the mapping $d_\alpha \mapsto b_\alpha$ defines an elementary map of N . By homogeneity this mapping has an extension to a total automorphism G of N . The required \bar{b} is $G\bar{b}^*$. Since $F + \langle \bar{a}, \bar{b} \rangle$ coincides with GH it is elementary.

6.2.16 A classification of homogeneous models.

Homogeneous models are determined up-to-isomorphism by their cardinality and the parameter-free types they realize.

PROPOSITION Let M and N be two homogeneous models with the same cardinality. Then every back-and-forth extendible elementary map $F : M \rightarrow N$ of smaller cardinality extends to an isomorphism $H : M \rightarrow N$. In particular, if M and N realize the same parameter-free types, then M and N are isomorphic.

PROOF We construct $H : M \rightarrow N$ as union of a chain of elementary maps F_α back-and-forth extendible between M and N . The chain starts with F , the limit stages are clear since Proposition 6.2.15 asserts that it is safe to take the union. At successor stages we use a back-and-forth strategy to make the limit of F_α total and surjective. We need only guarantee that $F_{\alpha+1}$ is an elementary map then, by Proposition 6.2.14, this is an extendible map. This proves the main claim of the proposition. Finally, observe that if M and N realize the same parameter-free types then, by Proposition 6.2.4 above, the empty map $\emptyset : M \rightarrow N$ is an extendible elementary map.

6.2.17 Uniqueness of saturated models.

Proposition 6.1.8, that is, the uniqueness of saturated models, can now be derived as corollary of the uniqueness of homogeneous models. If M and N are saturated models of the same cardinality, then they are homogeneous and, since they realize every type, they satisfy the hypotheses of the theorem above, so they are isomorphic.

6.2.18 Saturated \leftrightarrow homogeneous and weakly saturated.

Only a small amount of saturation is required for a homogeneous model to be saturated. We say that M is **weakly saturated** if every parameter-free type is realized in M .

PROPOSITION For every model M the following are equivalent:

- 1 M is saturated; and

2 M is homogeneous and weakly saturated.

PROOF (If we assume that saturated models of cardinality $|M|$ exist the theorem below follows easily from 6.2.16 above.) Clearly only the direction **2** \Rightarrow **1** needs a proof. Reasoning by induction on the cardinality of $A \subseteq M$, we show that M realizes every type $p(x)$ over A . Suppose the claim holds for sets of parameters cardinality $< \lambda$. We prove the claim holds for A of cardinality λ . Let \bar{a} be a tuple that enumerates A . We can write an arbitrary A -type as $q(\bar{a}x)$ where $q(\bar{y}x)$ is parameter-free. As every type over $< \lambda$ parameters is realized in M , by the argument in the proof of 4.3.2 we conclude that every finitely consistent set of parameter-free formulas with $\leq \lambda$ free variables is realized in M . Let $\bar{a}', b \subseteq M$ be a realization of $q(\bar{y}x)$. As $\bar{a} \equiv \bar{a}'$, by homogeneity, there is an automorphism $F : M \rightarrow M$ such that $F\bar{a}' = \bar{a}$. Then Fb realizes $q(\bar{a}x)$.

6.2.19 Homogeneous sets.

We say that a set A is (elementarily) homogeneous if every elementary map F with range and domain contained in A and of cardinality $< |A|$ has an extension to an elementary permutation of A (that is, an elementary map such that $F[A] = A$). We say that A is (elementarily) ω -homogeneous if the same holds for every finite F . Let A be an ω -homogeneous subset of \mathcal{U} . Suppose that every parameter-free type $p(\bar{x})$ has a realization in A . Prove that A is a model. **EXERCISE**

6.2.20 An exercise.

Prove that if M and N are homogeneous models such that M embeds elementary into N and N embeds elementary into M then they are isomorphic. Show that the hypothesis of homogeneity is essential (e.g., use dense linear orders). **EXERCISE**

6.2.21 An exercise.

Prove that M is homogeneous if and only if for every set $A \subseteq M$ of cardinality $< |M|$, if $a \equiv_A b$ then there is a A -automorphism of M which takes a to b . **EXERCISE**

Chapter 7

Omitting elements

The theme of this chapter is: construct a model containing some elements and omitting some other. We examine three different cases: when the set to be omitted is small, definable, or type-definable. Very important concepts will emerge e.g. that of *algebraic* elements and of *isolated* types.

Notation and global assumptions.

As in Chapter 6 above.

7.1 Omitting small sets

For given sets A and B we ask when there is a model M containing A but disjoint from B . A satisfactory answer can be given using the notion of algebraicity: a concept which is fundamental in all what follows.

7.1.1 Omitting a set.

When B a set and M a model, we say that M **omits** B if M is disjoint from B , that is, if $M \cap B = \emptyset$.

7.1.2 Algebraic elements.

We say that the formula $\varphi(\bar{x})$ is **algebraic** if it has finitely many solutions, that is, if $\exists^{=n} \bar{x} \varphi(\bar{x})$ holds for some positive integer n . (Recall that the expression $\exists^{=n} \bar{x} \varphi(\bar{x})$ is an abbreviation of the sentence saying that there are exactly n elements satisfying $\varphi(\bar{x})$; cf. Definition 2.2.10 above.) If $\exists^{=n} \bar{x} \varphi(\bar{x})$ holds in the universe then, by elementarity, it holds in every model M containing the parameters of $\varphi(\bar{x})$. Again by elementarity, tuples satisfying $\varphi(\bar{x})$ in one model satisfy $\varphi(\bar{x})$ in the universe.

It follows that the n tuples $\bar{b}_1 \cdots \bar{b}_n$ that satisfy $\varphi(\bar{x})$ all belong to every model M containing the parameters of $\varphi(\bar{x})$: none of these tuples can be omitted. If \bar{b} satisfies an algebraic A -formula we say that **\bar{b} is algebraic over A** . If \bar{a} and \bar{b} are such that \bar{a} is algebraic over \bar{b} and \bar{b} is algebraic over \bar{a} then we say that **\bar{a} and \bar{b} are interalgebraic**.

7.1.3 Algebraic closure.

We denote by **acl A** the set of elements algebraic over A , that is, the elements b such that $\varphi(b)$ holds for some algebraic formula. We call **acl A** the algebraic closure of A . A set A is said to be **algebraically closed** if $A = \text{acl } A$. Fortunately, we need not define the set of algebraic tuples: by Exercise 7.1.8 a tuple \bar{b} is algebraic over A if and only if $\bar{b} \subseteq \text{acl } A$. Note that **acl A** is closed under $f^{\mathcal{U}}$ (the interpretation of f in \mathcal{U}) for every function of the language. So **acl A** is a substructure of the universe but, in general, **acl A** is not a model (cf. Exercise 7.1.12 below).

7.1.4 Definable elements and definable closure.

We say that the tuple \bar{b} is **definable over A** , if there is an A -formula such that $\varphi(\bar{b})$ and $\exists^{-1} \bar{x} \varphi(\bar{x})$. We write **dcl A** for the set of elements definable over A . We call **dcl A** the **definability closure of A** . Again, we need not define the set of definable tuples: by Exercise 7.1.9 a tuple \bar{b} is definable over A if and only if $\bar{b} \subseteq \text{dcl } A$. If \bar{a} and \bar{b} are such that \bar{a} is definable over \bar{b} and \bar{b} is definable over \bar{a} then we say that **\bar{a} and \bar{b} are interdefinable**.

7.1.5 Almost-in.

When b belongs to M for every model M containing A , then we say that b is **almost-in A** . The term is non standard but we have coined it using a standard way of generating terminology: whenever a property holds for every model containing A we say it holds *almost* for A . In the next sections we shall encounter sets *almost-definable* over A , formulas *almost-satisfied* in A . By the argument in the paragraph above, algebraic tuples cannot be omitted: they are almost-in A . It is not difficult to show that the converse holds: a tuple that cannot be omitted is algebraic.

7.1.6 Algebraic \leftrightarrow almost-in.

From the theorem we now prove it follows that **acl A** is the intersection of all models containing A . This proof illustrates in a simple case how to build models enumerating its elements one-by-one; it not the shortest proof: in 7.1.13 below, we reprove this theorem using automorphisms.

PROPOSITION For every set A and b the following are equivalent:

1 $b \notin \text{acl } A$; and

2 there is model containing A that omits b .

PROOF The implication **2** \Rightarrow **1** has been discussed above. To prove that **1** \Rightarrow **2** we assume that $b \notin \text{acl } A$ and construct a model $M \supseteq A$ that omits b . The model M is the union of the chain: $A_0 = A$; $A_{\alpha+1} = A_\alpha + a_\alpha$; and union at limit stages. As in the proof of the Löwenheim-Skolem theorem, the elements a_α are chosen to satisfy a consistent formula $\psi(x)$ with parameters in A_α which is not satisfied in A_α . We require that $b \notin \text{acl } A_\alpha$. This is obviously guaranteed at limit stages, so the only thing we need to check is that at successor stages we can find a_α that meets this requirement.

CLAIM Assume $b \notin \text{acl } A$. Then every consistent A -formula $\psi(x)$ is satisfied by some a such that $b \notin \text{acl}(A + a)$.

PROOF Fix any complete A -type $p(x)$ containing $\varphi(x)$. It suffices to choose any a that realizes $p(x) + q(x)$ where

$$q(x) := \left\{ \neg\varphi(xb) : \begin{array}{l} \text{for every } A\text{-formula } \varphi(xz) \text{ such that} \\ \exists =^n z \varphi(xz) \in p(x) \text{ for some } n \in \omega \end{array} \right\}.$$

In fact, let a be any realization of $p(x) + q(x)$ and suppose for a contradiction that $b \in \text{acl}(A + a)$. Then $\varphi(ab)$ holds for some A -formula $\varphi(x, z)$ such that $\varphi(a, z)$ is algebraic. Since $p(x)$ is complete, $\exists =^n z \varphi(xz) \in p(x)$ for some n . Then the formula $\neg\varphi(x, b)$ is in $q(x)$ so $\neg\varphi(a, b)$, a contradiction.

We need only show that $p(x) + q(x)$ is finitely consistent. But first it is convenient to check that $q(x)$ is closed under conjunction. So, let $\neg\varphi_1(xb)$ and $\neg\varphi_2(xb)$ be two formulas in $q(x)$. Then $\exists =^n z \varphi_1(xz)$ and $\exists =^m z \varphi_2(xz)$ are both in $p(x)$. So, for some $k \leq m + n$, the formula

$$\exists =^k z \left[\varphi_1(xz) \vee \varphi_2(xz) \right]$$

is in $p(x)$. It follows that the conjunction of $\neg\varphi_1(xb)$ and $\neg\varphi_2(xb)$ is in $q(x)$ as required. Now, suppose for a contradiction that $p(x) + q(x)$ is inconsistent. Then for some formula $\vartheta(x)$ in $p(x)$ and some formula $\neg\varphi(xb)$ in $q(x)$,

$$\forall x \left[\vartheta(x) \rightarrow \varphi(xb) \right].$$

Since $p(x)$ is complete, we can assume $\vartheta(x)$ implies $\exists =^n z \varphi(xz)$ for some n , hence

$$\forall x \left[\vartheta(x) \rightarrow \varphi(xz) \right]$$

has at most n solutions. Then b is algebraic over A . This contradicts the hypothesis, so the claim is proved and with it the theorem.

7.1.7 Omitting a small set.

We show that we can omit a small set B if as soon as we can omit separately all its elements. For an alternative proof see Exercise 7.1.14 below.

PROPOSITION For every set A and B the following are equivalent:

1 $B \cap \text{acl } A = \emptyset$; and

2 there is model containing A that omits B .

PROOF The construction is similar to 7.1.6 above but the proof of the corresponding claim is more involved. The claim is proved assuming as induction hypothesis that the theorem holds for every set A and B' where B' of cardinality smaller than $|B|$. When B is finite we also assume: ***** there are no $b, c \in B$ such that $c \in \text{acl}(A + b)$. This is no loss of generality in fact, if $b, c \in B$ like above exist, then any model that omits $B \setminus c$ omits also b so the theorem follows directly from the induction hypothesis.

CLAIM If $B \cap \text{acl } A = \emptyset$. Then every consistent A -formula $\psi(x)$ is satisfied by some a such that $B \cap \text{acl}(A + a) = \emptyset$.

PROOF Fix any complete A -type $p(x)$ containing $\varphi(x)$. It suffices to choose any a that realizes $p(x) + q(x)$ where

$$q(x) := \left\{ \neg\varphi(xb) : \begin{array}{l} \text{for every } b \in B \text{ and every } A\text{-formula } \varphi(xz) \\ \text{such that } \exists =^n z \varphi(xz) \in p(x) \text{ for some } n \in \omega \end{array} \right\}.$$

It suffices to show that $p(x) + q(x)$ is finitely consistent. (Note that here we use that B has small cardinality, the same argument would not work for the larger sets considered in 7.2.3 below.) Suppose not, then for every x

$$\vartheta(x) \rightarrow \bigvee_{i=0}^n \varphi_i(xb_i)$$

for some formula $\vartheta(x)$ in $p(x)$ and some finite set of formulas $\varphi_i(x, b_i)$ in $q(x)$. Since $p(x)$ is complete we can also assume that for every x

$$\vartheta(x) \rightarrow \exists =^{n_i} z \varphi_i(xz).$$

If B is infinite the contradiction is immediate: no model containing A omits $\{b_0, \dots, b_n\}$, contrary to the induction hypothesis. When B is finite observe that for every x

$$\vartheta(x) \wedge \neg\varphi_0(xb_0) \rightarrow \bigvee_{i=1}^n \varphi_i(xb_i)$$

so, every model containing $A + b_0$ does not omit the set $\{b_1, \dots, b_n\}$. By induction hypothesis one of b_1, \dots, b_n is algebraic over $A + b_0$. This contradicts ***** above.

7.1.8 Transitivity and compositionality of algebraicity.

The following proposition proves some fundamental properties of algebraicity. Note that the equivalence $1 \Leftrightarrow 4$ amounts to $\text{acl}(\text{acl } A) = \text{acl } A$.

PROPOSITION Assume that \bar{b} is algebraic over A . The following are equivalent:

1 \bar{a} is algebraic over A

2 \bar{a} is algebraic over a finite subset of A ;

3 each element of the tuple \bar{a} is algebraic over A ; and

4 \bar{a} is algebraic over $A + \bar{b}$.

PROOF The equivalences **1** \Leftrightarrow **2** and **1** \Leftrightarrow **3** are straightforward. The equivalence **1** \Leftrightarrow **4** has an easy syntactic proof but it can also be derived from the theorem above: $\text{acl}(\text{acl} A)$ is the intersection of all models containing $\text{acl} A$ and this is the same as the intersection of all models containing A .

7.1.9 Transitivity and compositionality of definability.

The equivalences above hold also for definability, again the equivalence **1** \Leftrightarrow **4** amounts to $\text{dcl}(\text{dcl} A) = \text{dcl} A$.

PROPOSITION Assume that \bar{b} is definable over A . The following are equivalent:

1 \bar{a} is definable over A

2 \bar{a} is definable over a finite subset of A ;

3 each element of of the tuple \bar{a} is definable over A ; and

4 \bar{a} is definable over $A + \bar{b}$.

PROOF Again, **1** \Leftrightarrow **2** and **1** \Leftrightarrow **3** are straightforward. The equivalence **1** \Leftrightarrow **4** need now be proved syntactically. This is left to the reader as an exercise. **EXERCISE**

7.1.10 A remark.

The following property breaks the analogy between the algebraic and definability closure

$$\bar{a} \equiv_A \bar{b} \text{ then } \bar{a} \equiv_{\text{dcl} A} \bar{b}.$$

The reader can prove this as an exercise. **EXERCISE**

7.1.11 On the intersection of two models.

We have seen that the algebraic closure of A is the intersection of all models containing A . Actually the intersection of two models is already sufficient to get $\text{acl} A$. The proof of this fact is a slight variation of the construction above. For an alternative, shorter proof see Exercise 7.1.14 below.

PROPOSITION Let $A \subseteq N$ be arbitrary. There is an M such that $\text{acl} A = N \cap M$.

PROOF We construct M which intersects N only in $\text{acl} A$. The model M is the union of the chain: $C_0 = A$; $C_{\alpha+1} = C_\alpha + c_\alpha$; and union at limit stages. As above, the elements c_α are chosen to satisfy a consistent formula with parameters in C_α which is not satisfied in C_α . Since $A \subseteq M$ it suffices to ensure that $N \cap M \subseteq \text{acl} A$. The requirement $N \cap C_\alpha \subseteq \text{acl} A$ seems natural but it is not enough to keep induction going. So we require that $N \cap \text{acl} C_\alpha \subseteq \text{acl} A$. The only fact we need to check is that we can find c_α that meets this requirements. This is shown in the following claim (with C for C_α and c for C_α).

CLAIM Let C be such that $N \cap \text{acl} C \subseteq \text{acl}(A)$. Every consistent C -formula $\psi(x)$ is satisfied by some c such that $N \cap \text{acl}(C + c) \subseteq \text{acl} A$.

PROOF Let $p(x)$ be any complete C -type containing $\psi(x)$. It suffices to choose any c that realizes $p(x) + q(x)$ where

$$q(x) := \left\{ \begin{array}{l} \neg\varphi(xb) \quad : \quad \text{for } b \in N \setminus \text{acl} A \text{ and } \varphi(xy) \text{ any } C\text{-formula} \\ \text{such that } \exists^{\neq n} z \varphi(xz) \in p(x) \text{ for some } n \end{array} \right\}.$$

In fact, if $N \cap \text{acl}(C + c)$ contains some realization c of $p(x) + q(x)$, then $\varphi(cb)$ holds for some C -formula $\varphi(x, z)$ such that $\varphi(c, z)$ is algebraic. Then $\exists^{\neq n} z \varphi(xz) \in p(x)$ for some n . Since $b \in N$ but the formula $\neg\varphi(x, b)$ is not in $q(x)$ we conclude that $b \in \text{acl} A$ as required.

Now to show that $p(x) + q(x)$ is consistent reason as in the proof of the claim in 7.1.7 above.

7.1.12 Algebraic witnesses.

We say that T has algebraic witnesses (this is non-standard terminology) if for every A , every consistent A -formula $\varphi(x)$ has a witness in $\text{acl} A$. The reader can prove as an exercise that the following are equivalent:

1 T has algebraic witnesses;

2 models are closed under intersection; and

3 $\text{acl} A$ is a model for every A .

(Another useful characterization is given in 13.1.3 below) Note that $\varphi(x)$ above has just one free variable: this makes it easier to check that a theory has algebraic witnesses. E.g. vector spaces, algebraically closed fields, real closed fields, and Peano arithmetic have algebraic witnesses (the first three claims require elimination of quantifiers, the last one follows from the least number principle). **EXERCISE**

7.1.13 Algebraic \leftrightarrow finite orbit \leftrightarrow almost-in.

We sketch a second proof of Proposition 7.1.6 above. The proof is based on many facts we have learned in the previous chapters. We obtain, as a spin-off, another semantic characterization of algebraicity. Recall that orbits have been defined in 6.2.11 above.

PROPOSITION For A and b arbitrary, the following are equivalent:

1 b is almost-in A ;

2 $O_A(b)$ is finite; and

3 $b \in \text{acl} A$.

PROOF To prove **-2** \Rightarrow **-1** fix some arbitrary model $M \supseteq A$. Suppose $O_A(b)$ is infinite, so, it has cardinality κ . So, by obvious cardinality reasons, there is an A -automorphism F such that $Fb \notin M$, so, $b \notin F^{-1}[M]$. Observe that $F^{-1}[M]$ is a

model (see Exercise 3.2.5 above). So some model omits b . To prove the implication $2 \Rightarrow 3$ assume that $O_A(b)$ is finite. Since $O_A(b)$ equals $p(\mathcal{U})$ where $p(x)$ is the type of \bar{b} over A , we have that $p(x)$ is algebraic, that is, it has finitely many realizations. So b is algebraic (see Exercise 4.4.10 above).

7.1.14 An exercise.

Use the method of 7.1.13 to prove Proposition 7.1.7 and 7.1.11 above. (Hint: consider property $(*)$ in 7.1.7 above.) EXERCISE

7.1.15 Definable \leftrightarrow fixed point.

For easy of reference we state the following special case of the proposition above. The proof should be clear.

PROPOSITION For A and \bar{b} arbitrary, the following are equivalent:

- 1 \bar{b} is fixed by every A -automorphism; and
- 2 \bar{b} is definable over A .

7.1.16 An exercise: definable sets of algebraic elements are finite.

This exercise can be solved either directly, or applying Proposition 7.1.6, or applying Exercise 4.4.6 above. Show that if all the solutions of $\varphi(\bar{x})$ are algebraic over A then $\varphi(\bar{x})$ is algebraic. In other words, if $\varphi(\bar{\mathcal{U}})$ is infinite, then it contains some tuple which is not algebraic over A . EXERCISE

7.1.17 An exercise.

Let F be an automorphism of the universe. Recall that $F[A]$ denotes the image of A under F . Prove that $F[\text{acl}(A)] = \text{acl}(F[A])$ for every set A . EXERCISE

7.2 Omitting definable sets

The characterization of almost-satisfiability that we prove in this section has some application in next chapters. But the main reason to include it here is that it illustrates an application of the (elementary) diagram method.

7.2.1 Almost-satisfiability.

Given a set A , and a formula $\varphi(\bar{x})$ (with arbitrary parameters), we say that M **omits** $\varphi(\bar{x})$ if M omits $\varphi(\bar{\mathcal{U}})$, in other words, if $\varphi(M) = \emptyset$. We ask when there is a model $M \supseteq A$ that omits $\varphi(\bar{x})$. Note that the question is non-trivial only when

$\varphi(\bar{x})$ depends on parameters not in A , in fact, by elementarity, every model $M \supseteq A$ satisfies every consistent A -formula. If every model $M \supseteq A$ satisfies $\varphi(\bar{x})$, then we say that $\varphi(\bar{x})$ is **almost-satisfied in A** .

7.2.2 The diagram method.

In the previous chapter we have constructed models by looking one-by-one for suitable elements, here we apply elementary diagram method. The diagram lemma 3.2.11 and Exercise 3.2.4 are facts we need for the following proposition.

7.2.3 A characterization of almost-satisfiability.

This characterization is one of the basic tools we need in Sections ?? and ?? below. Note that a straightforward sufficient condition for an formula $\varphi(\bar{x})$ to be almost-satisfied in A is the existence of some A -formula $\vartheta(\bar{x})$ such that $\vartheta(\bar{x}) \rightarrow \varphi(\bar{x})$. In general, this is not a necessary condition: for instance let T be the theory of the random graph, or the theory of the linear order without endpoints. The formula $x \neq b$ is almost-satisfied by \emptyset (because all models are infinite) but no consistent parameter-free formula $\vartheta(x)$ is such that $\vartheta(x) \rightarrow x \neq b$ because in a dense linear order without endpoints all elements have the same type over \emptyset . So $\mathcal{U} \setminus b$ has no \emptyset -definable subset, still note we can come close to this: the set $\mathcal{U}^2 \setminus \langle b, b \rangle$ has a \emptyset -definable subset. Indeed this is quite obvious: the set defined by formula $x_1 \neq x_2$ will do. This is an example of general fact.

PROPOSITION Let A be a set and let $\varphi(\bar{x})$ be an arbitrary formula. The following are equivalent:

- 1 A almost-satisfies $\varphi(\bar{x})$; and
- 2 there is a consistent A -formula $\vartheta(\bar{u}_1 \cdots \bar{u}_n)$ such that for every $\bar{u}_1 \cdots \bar{u}_n$

$$\vartheta(\bar{u}_1 \cdots \bar{u}_n) \rightarrow \bigvee_{i=1}^n \varphi(\bar{u}_i).$$

PROOF Only the implication $1 \Rightarrow 2$ needs a proof. Let $N \supseteq A$ be arbitrary. Let $p(\bar{x})$ be the elementary diagram of N over A . (The elementary diagram of a model has been defined in 3.2.10 above). Let $q(\bar{x})$ be the set of formulas $p(\bar{x}) + \{\neg\varphi(\bar{u}) : \bar{u} \subseteq \bar{x}\}$. By the diagram lemma, the range of a realization of $p(\bar{x})$ is a model containing A . This model clearly omits $\varphi(\bar{u})$. So $q(\bar{x})$ is inconsistent. Then for some conjunction $\vartheta(\bar{u}_1 \cdots \bar{u}_n)$ of formulas in $p(\bar{x})$ we have that $\vartheta(\bar{u}_1 \cdots \bar{u}_n) + \neg\varphi(\bar{u}_1) \cdots \neg\varphi(\bar{u}_n)$ is inconsistent. So we obtain 2 as required.

7.2.4 An exercise.

Let $\varphi(x)$ be an arbitrary formula. Prove that, if $\varphi(\mathcal{U})$ contains a model (which may not contain the parameters of the formula), then it contains models of arbitrary

cardinality.

EXERCISE

7.3 Omitting type-definable sets

The theorem we prove here is a very classical basic result but it is not a result specific of model theory. Its nature lies more in general topology, specifically, Baire category theory. For this reason we add a few comments on that. These are not necessary to understand the theorem, so the reader less familiar with topology may simply skip them.

7.3.1 Type-definable sets.

Recall that a set is **type-definable** if it is of the form $p(\bar{U})$ for some type $p(\bar{x})$. Given an A -type $p(\bar{x})$, we ask if there is a model $M \supseteq A$ that omits $p(\bar{x})$. The answer is non-trivial: a model M as required may not exist. Its existence depends (in some cases) on a property which we now define.

7.3.2 Isolated types.

Let $p(\bar{x})$ be a type and let $\varphi(\bar{x})$ a formula. We say that **$\varphi(\bar{x})$ isolates $p(\bar{x})$** if $\varphi(\bar{x}) \rightarrow \psi(\bar{x})$ holds for every $\psi \in p$. In other words, if we have that $\varphi(\bar{U}) \subseteq p(\bar{U})$. We say that the a set **A isolates $p(\bar{x})$** if some A -formula isolates $p(\bar{x})$. When A is clear from the context we shall simply say **$p(\bar{x})$ is isolated**. E.g. when $p(\bar{x})$ is declared to be an A -type, *isolated* means that A isolates $p(\bar{x})$ unless otherwise stated.

7.3.3 A topology on the set of types.

The term *isolated* comes from topology. Write $S_n(A)$ for the set of complete A -types $p(\bar{x})$ where \bar{x} has arity n . Modulo the identification of types with filters of the Boolean algebra $\text{Def}_A \mathcal{U}$, complete types corresponds to ultrafilters. It is entirely classical to equip the set of ultrafilters on a Boolean algebra with the structure of a topological space. We spell this out in this special case. For every A -formula let $[\varphi(\bar{x})]$ denote the set of those types in $S_n(A)$ that contain $\varphi(\bar{x})$. By definition, a base of closed sets of $S_n(A)$ is formed by the sets $[\varphi(\bar{x})]$. This base is closed under union and intersection. It is also closed under complementation, so its elements are both closed and open (for short *clopen*). A complete A -type is isolated by an A -formula if and only if it is the unique element of some clopen set $[\varphi(\bar{x})]$. That is, if it is isolated point in the topological sense. The reader may wish to check that the topological space $S_n(A)$ is compact and all clopen sets of $S_n(A)$ are of the form $[\varphi(\bar{x})]$ for some A -formula.

7.3.4 An exercise.

Let $p(\bar{x})$ be a complete A -type. Let L^+ be an expansion of L with a new relation r . Let \mathcal{U}^+ be the expansion of \mathcal{U} that interprets r in the relation $p(\bar{U})$. Prove that \mathcal{U}^+ is saturated if and only if $p(\bar{x})$ is isolated.

EXERCISE

7.3.5 Algebraic types are isolated.

We say that a type is algebraic if it has finitely many realizations. You can prove as an exercise that if an A -type $p(\bar{x})$ is algebraic then A isolates $p(\bar{x})$.

EXERCISE

7.3.6 An exercise.

Let A be a set and let \bar{b} be a tuple. You can prove as exercise that \bar{b} is algebraic over A if and only if the B -type of \bar{b} is isolated for every $B \supseteq A$.

EXERCISE

7.3.7 Omitting types theorem.

Observe that the notions M isolates $p(\bar{x})$ and M realizes $p(\bar{x})$ coincide when M is a model. In fact, if \bar{a} realizes $p(\bar{x})$ then the formula $\bar{x} = \bar{a}$ trivially isolates $p(\bar{x})$ and, vice versa, if $\varphi(\bar{x})$ isolates $p(\bar{x})$ then every witness of $\varphi(\bar{x})$ realizes $p(\bar{x})$. So if A isolates $p(\bar{x})$, then every model containing A realizes $p(\bar{x})$. Below we prove a classical theorem: (when L and A are countable) the converse is also true. The construction proceeds by stages. Along the construction we preserve the property of being *isolated* since that of being *realized* is not well behaved in sets which are not models. In the limit, i.e. when a model is obtained, these two properties coincide.

PROPOSITION Let L and A be countable. Let $p(\bar{x})$ be a type over A . The following are equivalent:

- 1 there is a model $M \supseteq A$ that omits $p(\bar{x})$; and
- 2 A does not isolate $p(\bar{x})$.

PROOF The direction from 1 to 2 has been discussed above. To prove the converse we construct a model M as union of the chain: $A_0 = A$; $A_{\alpha+1} = A_\alpha + a_\alpha$; and union at limit stages. As usual the α -th element a_α is chosen to satisfy some consistent formula $\psi(y)$ with parameters in A_α following the strategy explained in 3.3.3 above. We also require that A_α does not isolate $p(\bar{x})$. Then clearly the limit M does not isolate $p(\bar{x})$. Note that since LA is assumed to be countable, we can assume that α is countable (or even finite) so A_α is countable for every α . In the claim below we prove that we can find a_α which preserves the induction hypothesis: $A_\alpha + a_\alpha$ does not isolate $p(\bar{x})$.

CLAIM Let LA be countable. Let $p(\bar{x})$ be an A -type which is not isolated by A . Every consistent A -formula $\psi(y)$ is satisfied by some a such that $A + a$ does not isolate $p(\bar{x})$.

PROOF We construct an A -type $q(y)$ containing $\psi(y)$. From the construction it will be clear that any a realizing $q(y)$ is as required by the claim. We let $q(y)$ be the union of the finite types $q_i(y)$ which we now define. Let $\xi_i(\bar{x}y)$ for $i \in \omega$ be an enumeration of all A -formulas with free variables among $\bar{x}y$. Below we need an enumeration of length ω ; here we use that LA is countable. Let $q_0(y)$ contain just $\psi(y)$. At stage $i+1$, assume that $\xi_i(\bar{x}y)$ is consistent with $q_i(y)$, otherwise let $q_{i+1}(y)$ be $q_i(y)$. Then we let

$$q_{i+1}(y) = q_i(y) + \exists \bar{x} \left[\xi_i(\bar{x}y) \wedge \neg \varphi(\bar{x}) \right]$$

for some arbitrary A -formula $\varphi(\bar{x})$ in $p(\bar{x})$ that leaves $q_{i+1}(y)$ consistent. This move ensures that $\xi_i(\bar{x}a)$ for a satisfying $q(y)$ does not isolate $p(\bar{x})$ so, in the end, that no $(A+a)$ -formula isolates $p(\bar{x})$. The proof is complete if we can show that we can always find a formula $\varphi(\bar{x})$ as required above. So suppose by way of contradiction that no formula makes $q_{i+1}(y)$ consistent, that is, for every $\varphi(\bar{x})$ in $p(\bar{x})$ we have

$$\xi_i(\bar{x}y) \wedge \bigwedge q_i(y) \rightarrow \varphi(\bar{x}).$$

Note that this is a well-formed formula because $q_i(y)$ is finite. By logic, we have that for every $\varphi(\bar{x})$ in $p(\bar{x})$

$$\exists y \left[\xi_i(\bar{x}y) \wedge \bigwedge q_i(y) \right] \rightarrow \varphi(\bar{x}).$$

This means that $p(\bar{x})$ is isolated by an A -formula, a contradiction.

7.3.8 Omitting meager sets of types.

Readers with an inclination for point-set topology may reflect on the analogies between the omitting type theorem above and the Baire category theorem (which says that every non-meager subset of the real line is non-empty). The following can be proven as exercise. The reader can prove that given any meager set of types $P \subseteq S_n(A)$ there is a model that omits every type in P . EXERCISE

Chapter 8

Poor man's geometry

There is a very natural class of first-order theories with a very strong geometric flavor. With the techniques introduced so far we can sketch a few basic properties of these theories.

Notation and global assumptions.

As in Chapter 6 above. Moreover in this chapter T is always assumed to be strongly minimal, see definition below (but we may restate it in the main theorems for emphasis). In Section 8.3 we further assume that $\text{acl } \emptyset$ is infinite.

8.1 Strongly minimal theories

A theory is strongly minimal if the definable sets (of arity one) are as simple as they could possibly be: finite or cofinite (i.e. the complement is a finite set). Affine spaces and algebraically closed fields are the canonical examples. In this section we classify the models of a complete strongly minimal theory. We will also see that these theories are uncountably categorical so this result is, in a way, a particular case of Morley's theorem and Baldwin-Lachlan's theorem. Indeed the results below establish the relatively easy algebraic part of these theorems. A deeper model theoretical analysis is necessary to obtain the full theorems: eventually we will see that in the core of every model of an uncountably categorical theory there is a strongly minimal structure and we will see how this structure relates to the whole model.

8.1.1 Strongly minimal theories.

We say that T is **strongly minimal** if every definable set is finite or cofinite. That is, for every formula $\varphi(x)$ either $\varphi(\mathcal{U})$ or $\neg\varphi(\mathcal{U})$ is a finite set (it is important to note that $\varphi(x)$ depends on one single variable, definable sets of tuples of larger arity

can still be rather complex). By the saturation of \mathcal{U} (cf. Exercise 4.4.9 above) this amounts to requiring that for every parameter-free formula $\varphi(\bar{z}x)$ there is an n such that

$$\forall \bar{z} \left[\exists^{<n} x \varphi(\bar{z}x) \vee \exists^{<n} x \neg\varphi(\bar{z}x) \right].$$

This shows that if T is strongly minimal then for every model M all subsets of M definable over M are either finite or cofinite in M .

8.1.2 Examples.

The theory of equality, vector spaces, and algebraically closed fields are strongly minimal. This follows immediately from elimination of quantifiers. A theory with elimination of quantifiers is strongly minimal as soon as the atomic sentences with at most one free variable define finite (or cofinite) sets. In the empty language the atomic formulas with one free variable have the form $a = x$ for some parameter a . In the theory of vector spaces they have (up to equivalence) the form $kx = a$ for some element k of the field. In the theory of algebraically closed fields atomic formulas may be reduced to those of the form $t(x) = 0$ for some polynomial $t(x)$. Then these theories are easily seen to be strongly minimal.

8.1.3 Strongly minimal sets.

What we say in this section about strongly minimal theories actually holds for the more general notions of strongly minimal set. Let \mathcal{D} be a definable set, that is, a set of the form $\delta(\bar{\mathcal{U}})$ for some formula $\delta(\bar{x})$. We say that \mathcal{D} is a **strongly minimal set** or that $\delta(\bar{x})$ is a **strongly minimal formula** if \mathcal{D} is infinite and for every formula $\varphi(\bar{x})$ either $\varphi(\mathcal{D})$ or $\mathcal{D} \setminus \varphi(\mathcal{D})$ is finite. (This definition does not assume T to be strongly minimal.) Again we have that the saturation \mathcal{U} implies that a bound on the cardinality of $\varphi(\mathcal{D})$, casus quo, $\mathcal{D} \setminus \varphi(\mathcal{D})$, may be found uniformly in the parameters of $\varphi(\bar{x})$. Observe that a theory T is strongly minimal if and only if the formula $x = x$ is strongly minimal.

It is important to note that the parameters of the formula $\varphi(\bar{x})$ above range over the whole universe not only over \mathcal{D} . Actually, it is useful to observe that we may restrict them to range over any saturated model containing the parameters of $\delta(\bar{x})$. The proof is left to the reader as an exercise.

PROPOSITION Let N be any saturated model and let \mathcal{D} be definable over N . The following are equivalent

- 1 \mathcal{D} is strongly minimal; and
- 2 for every N -formula $\varphi(\bar{x})$, either $\varphi(\mathcal{D})$ or $\mathcal{D} \setminus \varphi(\mathcal{D})$ is finite.

EXERCISE

8.1.4 A remark.

In the definition of strong minimality we could have used any ω -saturated model in place of \mathcal{U} and still obtain the syntactic characterization of T given above. But some saturation is essential. We say that M is a **minimal model** if all M -definable subsets of M are either finite or cofinite in M (do not confuse this with the homonymous notion defined in 9.1.20 below). The existence of a minimal model does not imply that T is strongly minimal. For instance let T be the theory of the natural numbers with the sole relation of order. The natural numbers are a minimal model (the verification of this assertion requires the proof of elimination of quantifiers for T , see Exercise 5.3.12 above) but clearly T is not strongly minimal: take a saturated model and a non-standard element (i.e. an element larger than all natural numbers), then the formula $x < a$ defines a set which is neither finite nor cofinite.

8.1.5 Examples.

There are three main examples of strongly minimal theories: the theory of equality, the theory of vector spaces, and the theory of algebraically closed fields. When specialized in the respective theories the notion of dimension defined below becomes: cardinality, linear dimension, degree of transcendence.

8.1.6 Classifications.

Assuming that T is strongly minimal, we shall classify the models of T . This illustrates, in an ideally simple situation, what we mean by classifying models of a complete theory. The example is really the most simple: the problem is in general very complex.

It needs to be emphasized that T is a complete theory (this has been assumed for whole chapter). One could relax this hypothesis and look for a classification of models of arbitrary strongly minimal theories. This is a much harder problem; it belongs to the realm of the so-called geometrical stability theory; the reader is referred to (the introductions of) [BOUSCAREN] and [PILLAY II].

8.1.7 There is at most one non-algebraic type (of arity one).

The following is a very important characterization of strongly minimal theories: the unique non-algebraic type over A is called **the transcendental type over A** . One immediate consequence of the following proposition is that strongly minimal theories are ω -stable (this is defined in ?? below). The proof is left to the reader as an exercise.

PROPOSITION Let A be an algebraically closed set. The following are equivalent:

- 1 T is strongly minimal; and
- 2 for every set A , every $a, b \notin A$ are A -elementary equivalent. EXERCISE

8.1.8 The operator of algebraic closure.

In a strongly minimal theory the operator of algebraic closure acl is a **finitary closure operator with the exchange property** that is it enjoys the following properties:

- 1 **finite nature**: if $a \in \text{acl } C$ then $a \in \text{acl } \bar{c}$ for some $\bar{c} \subseteq C$
- 2 **monotonicity**: $C \subseteq \text{acl } C$
- 3 **transitivity**: $\text{acl } C = \text{acl}(\text{acl } C)$
- 4 **symmetry**: if $a, b \notin \text{acl } C$ then $b \in \text{acl}(C + a)$ if and only if $a \in \text{acl}(C + b)$.

The last property is also called **Stainitz exchange principle**. Indeed, that acl is a finitary closure operator is true for every T (cf. Exercise 7.1.8 above). The exchange property requires strong minimality. The proof is given below.

8.1.9 Exchange property.

Eventually we shall see that this proposition is actually a corollary of a more general symmetry property: the exchange property of forking. This is interesting in itself; but a direct proof is actually easier.

PROPOSITION Let T be strongly minimal. For every C , a and b if $a \in \text{acl}(C + b) \setminus \text{acl } C$ then $b \in \text{acl}(C + a)$.

PROOF Assume $b \notin \text{acl}(C + a)$ and $a \in \text{acl}(C + b)$ we show that $a \in \text{acl } C$. Let $\varphi(x, y)$ be a C -formula such that $\varphi(b, a) \wedge \exists^{=n} y \varphi(b, y)$. Since $b \notin \text{acl}(C + a)$ then the formula $\varphi(x, a) \wedge \exists^{=n} y \varphi(x, y)$ has infinitely many solutions so, since T is strongly minimal, it holds for all but finitely many x . Then every model containing C contains some solution of $\varphi(x, a) \wedge \exists^{=n} y \varphi(x, y)$. But a is algebraic in any of these solutions. Hence every model containing C contains a . So $a \in \text{acl } C$.

8.1.10 Bases and dimension.

We say that a is **independent from B** if $a \notin \text{acl } B$. We say that B is an **(algebraically) independent set** if every $a \in B$ is independent from $B \setminus a$. We say that B is a **base for C** if B is an independent subset of C and $C \subseteq \text{acl } B$. Below we prove that all bases for C have the same cardinality. We call this cardinality the **dimension of C** and we denote this by **$\dim C$** .

8.1.11 Localization.

The definitions and theorems above can be relativized to some arbitrary set of parameters A . We write $\text{acl}_A C$ for $\text{acl}(AC)$. It is immediate to check that $\text{acl}_A C$ is a finitary closure operator with the exchange property. The definitions above localize to A : we say that a is *independent from B over A* , that B is an *independent set over A* , that B is a *base for C over A* . We write $\dim_A C$ for the *dimension of C over A* .

8.1.12 Bases and maximal independent subset.

We need the following standard characterization of bases; observe that the use of the exchange property is crucial.

PROPOSITION Let T be strongly minimal and let $B \subseteq C$ be arbitrary. The following are equivalent:

- 1 B is a base of C ;
- 2 B is a maximal independent subset of C .

PROOF One direction is clear: if no element of C is independent from B then B is maximal. For the converse, let $a \in C$, by maximality $B + a$ is not an independent set, so, either $a \in \text{acl} B$ or $b \in \text{acl}((B + a) \setminus b)$ for some $b \in B$. In the second case, since $b \notin \text{acl}(B \setminus b)$, by the exchange property, $a \in \text{acl} B$. This shows that $C \subseteq \text{acl} B$ so B is a base of C .

8.1.13 An observation.

The following is straightforward but worth of mention. Let T be strongly minimal. If C is a set cardinality larger than $|T|$ then $\dim C = |C|$.

8.1.14 All bases have the same cardinality.

The following proposition is the analogous to an elementary theorem of linear algebra; the proof follows the same lines.

PROPOSITION Let T be a strongly minimal theory. Let C and A be arbitrary. For every independent set $B_0 \subseteq C$ there is a bases B such that $B_0 \subseteq B \subseteq C$. Any two bases of C over A have the same cardinality. The same holds when localized to any arbitrary set A .

PROOF By the lemma of Zorn we now that there is a maximal set B such that $B_0 \subseteq B \subseteq C$ and B is independent over A (the finite character of the algebraic closure is used to show that the union of a chain of independent sets is independent). Now we prove that two bases have the same cardinality. Let B_1 and B_2 be two bases of C over A . Suppose for a contradiction that $|B_1| < |B_2|$. Suppose first that B_2 is infinite. For each element of $b \in B_1$ choose a $\bar{c} \subseteq B_2$ such that $b \in \text{acl} \bar{c}$; the union of

all these finite sets form a subset $D_2 \subseteq B_2$ such that $B_1 \subseteq \text{acl}_A D_2$ and $|D_2| < |B_2|$. Then $C \subseteq \text{acl}_A D_2$ and $B_2 \setminus D_2$ is non-empty. This contradicts the independence of B_2 . When B_2 is finite, we proceed by induction on the cardinality of B_2 . Let $c \in B_2$, and define $D_2 = B_2 \setminus c$; clearly D_2 is a bases of C over $A + c$. Using the first claim of this proposition we can find a $D_1 \subseteq B_1$ that is a base of C over $A + c$. Since $c \in \text{acl} B_1$ then B_1 is not independent over $A + c$. So $|D_1| < |B_1|$. Now observe that $|D_1| < |D_2|$ and apply the induction hypothesis to derive a contradiction.

8.1.15 An exercise.

Prove that $\dim(AB) - \dim A = \dim_A B$ holds for every A and B .

EXERCISE

8.1.16 Bases and isomorphisms.

Here we prove that when T is strongly minimal any bijection between algebraically independent sets is an elementary map. Recall Exercise 7.1.17 above (which holds in every theory T): if F is an automorphism and $F[B] = C$ then $F[\text{acl} B] = \text{acl} C$. So two models of a strongly minimal theory with the same dimension are isomorphic.

PROPOSITION Let T be a strongly minimal theory. If B and C are two A -independent sets of the same cardinality then every bijection between B and C is an A -elementary map.

PROOF We have to prove that $\varphi(\bar{b}) \leftrightarrow \varphi(\bar{c})$ for every A -formula $\varphi(\bar{x})$, every $\bar{b} \subseteq B$ and every $\bar{c} \subseteq C$. This is proved by induction on the arity of \bar{b} and \bar{c} . So, assume the claim is true for \bar{b} and \bar{c} and prove it for the tuples $\bar{b}b'$ and $\bar{c}c'$ where $b' \in B \setminus \bar{b}$ and $c' \in C \setminus \bar{c}$ are arbitrary. Let $\varphi(\bar{x}y)$ be an A -formula and let $d \in B \setminus \bar{b}$ be arbitrary. By induction hypothesis $\varphi(\bar{b}y)$ is algebraic if and only if $\varphi(\bar{c}y)$ is algebraic. In the first case, by the A -independence of B and C , we have that $\neg\varphi(\bar{b}b')$ and $\neg\varphi(\bar{c}c')$. In the second case $\varphi(\bar{b}b')$ and $\varphi(\bar{c}c')$.

8.1.17 A toy version of Morley and Baldwin-Lachlan's theorems.

We have the following corollary: two models of a strongly minimal theory T are isomorphic if and only if they have the same dimension. In particular, by what observed in 8.1.13 above, T is categorical in every cardinal $\lambda > |L|$. The same conclusion, but assuming only that T is categorical in some $\mu > |L|$, is a celebrated theorem of Michael Morley (for countable languages) and of Saharon Shelah (for arbitrary languages). Morley's theorem will be proved in Chapter ?? below. The proof we give (due to John Baldwin and Alistar Lachlan) uses essentially the results in this section. Now, suppose L is countable and reason again under the hypothesis that T is strongly minimal. Note that if M has finite dimension n then for every $b \notin M$ the set $\text{acl}(M + b)$ is a model of dimension $n + 1$ and hence non-isomorphic

to M . So either all countable models of T have infinite dimension so they are all isomorphic and T is countably categorical, or there are infinitely many non-isomorphic countable models. The same conclusion can be obtained simply assuming that T is categorical in some uncountable cardinal. This is the famous theorem of Baldwin and Lachlan.

8.1.18 An exercise.

Let T be strongly minimal. Prove that if $\text{acl } C$ is infinite then it is a model. So, the intersection of two models M and N is a model whenever it is infinite. EXERCISE

8.2 Independence

The give a definition of independence that is tailored to work in a strongly minimal theory. For a more general definition of independence we have to wait until the notion of (non)forking is introduced.

8.2.1 Independence and dimension

Let T be strongly minimal. If $\dim_A B = \dim_{AC} B$, we say that **B is independent of C over A** and write

$$B \downarrow_A C$$

When A and/or B are singletons this notion coincide with that of 8.1.11 above.

8.2.2 The symmetry of independence

The following is one of the most important properties of the relation of independence.

PROPOSITION Let T be strongly minimal. If $B \downarrow_A C$ then $C \downarrow_A B$.

PROOF By exercise 8.1.15 above,

$$\begin{aligned} B \downarrow_A C &\Leftrightarrow \dim_A B = \dim_A BC - \dim_A C \\ C \downarrow_A B &\Leftrightarrow \dim_A C = \dim_A BC - \dim_A B, \end{aligned}$$

hence they are equivalent.

8.2.3 Independence and finite satisfiability.

The following characterization of independence is used in the next section. It is interesting because it gives a notion of independence that is more general. Some-time property (2) below is expressed saying that the type of \bar{b} over $M\bar{a}$ is finitely

satisfiable in M . (The following proposition considers only independence *over a model* for independence *over a set* see the exercise below)

PROPOSITION Let T be strongly minimal. The following are equivalent

- 1 $\bar{a} \downarrow_M \bar{b}$; and
- 2 for every M -formula $\varphi(\bar{x}\bar{y})$ such that $\varphi(\bar{a}\bar{b})$ there is an $\bar{b}' \subseteq M$ such that $\varphi(\bar{a}\bar{b}')$.

PROOF We prove $1 \Rightarrow 2$ by induction on $\dim_M(\bar{b})$. If this is 0, then $\bar{b} \subseteq M$ and there is nothing to prove. Now assume as induction hypothesis that the claim holds for \bar{b} and fix b such that $\dim_M(\bar{b}b) = \dim_M \bar{b} + 1$. By hypothesis, $\dim_{M\bar{a}}(\bar{b}b) = \dim_{M\bar{a}} \bar{b} + 1$ so $\varphi(\bar{a}\bar{b}b)$ implies that $\varphi(\bar{a}\bar{b}\mathcal{U})$ is coinfinite. Then $\varphi(\bar{a}\bar{b}\mathcal{U})$ contains some $b' \in M$. Now apply the induction hypothesis to the formula $\varphi(\bar{a}\bar{b}b')$.

We show that (2) implies $\bar{b} \downarrow_M \bar{a}$, then $2 \Rightarrow 1$ follows by symmetry. We reason by induction on $\dim_M \bar{a}$. Suppose the claim is true for \bar{a} and prove it for $\bar{a}a'$ where $\dim_M(\bar{a}a) = \dim_M \bar{a} + 1$. We need to show that $\dim_{M\bar{b}}(\bar{a}a) = \dim_{M\bar{b}} \bar{a} + 1$. Suppose not, then $a \in \text{acl}_{M\bar{b}} \bar{a}$, so there is an M -formula such that $\psi(\bar{b}\bar{a}a) \wedge \exists^{<n} y \psi(\bar{b}\bar{a}y)$. By (2) we can replace \bar{b} with dome $\bar{b}' \subseteq M$ and obtain that $a \in \text{acl}_M \bar{a}$ a contradiction.

8.2.4 An exercise.

Prove that the following are equivalent:

- 1 $\bar{a} \downarrow_A \bar{b}$; and
- 2 for every model M containing A and for every M -formula $\varphi(\bar{x}\bar{y})$ such that $\varphi(\bar{a}\bar{b})$ there is an $\bar{b}' \subseteq M$ such that $\varphi(\bar{a}\bar{b}')$. EXERCISE

8.2.5 An exercise.

Let M and \bar{a} be arbitrary. Let $p(\bar{x})$ be a type over M . Then $p(\bar{x})$ is realized by some \bar{b} such that $\bar{a} \downarrow_M \bar{b}$. Prove that the same holds when $p(x)$ is a type over $M\bar{b}$, provided that it is finitely satisfied in M (that is, every formula in $p(\bar{x})$ has a solution in M). EXERCISE

8.2.6 An exercise.

Generalize the exercise above to show that if $A \downarrow_M B$ then there is a saturated model N such that $B \subseteq N$ and $A \downarrow_M N$. EXERCISE

8.3 Modularity and linearity

We want to show, at this early stage of the course, that strongly minimal theories are rich enough to develop some geometric ideas.

We cannot use \mathcal{U}^{eq} nor canonical bases yet so, to keep the proofs concise, we work under the assumption that $\text{acl } \emptyset$ is infinite (we want that T weakly eliminates imaginary, see Chapter 13.3). Necessarily, we define linearity without referring to canonical bases but the notion is the usual one.

8.3.1 Planar curves.

A **planar curve** is a strongly minimal subset of \mathcal{U}^2 . We say that two planar curves are **similar** if their symmetric difference is finite. By strong minimality a curve over A and a curve over B are similar if and only if they intersect outside of $\text{acl}(AB)$ if and only if they coincide outside of $\text{acl}(AB)$ (hence similarity is an equivalence relation).

8.3.2 A lemma.

This proposition is an easy corollary of a general fact about Morley rank which we will prove later. Here we give a direct proof.

PROPOSITION Let M and bc be such that $\dim_M(bc) = 1$. Then bc belongs to a planar curve over M .

PROOF Fix an M -formulas $\varphi(xy)$ such that

$$\varphi(bc) \wedge \exists^=n y \varphi(by)$$

for some n . Choose these formulas so that n is minimal. We claim that $\varphi(xy)$ is a planar curve. To prove the claim, we fix a saturated model N such that $bc \downarrow_M N$, which exists by exercise 8.2.6 above. We fix also an arbitrary M -formula $\psi(\bar{z}xy)$ and an arbitrary $\bar{d} \subseteq N$. We need to prove that either $\varphi(xy) \rightarrow \psi(\bar{d}xy)$ or $\varphi(xy) \rightarrow \neg\psi(\bar{d}xy)$ holds for every $x, y \notin N$. We assume $\psi(\bar{d}bc)$, the same argument apply when $\neg\psi(\bar{d}bc)$. It cannot be that $\exists y [\varphi(by) \wedge \neg\psi(\bar{d}by)]$ otherwise we could replace \bar{d} with some $\bar{d}' \subseteq M$ and contradict the minimality of n . So it must be that $\forall y [\varphi(by) \rightarrow \psi(\bar{d}by)]$ and since $b \notin N$ then $\forall y [\varphi(xy) \rightarrow \psi(\bar{d}xy)]$ holds for every $x \notin N$.

8.3.3 Linearity.

We say that **T is linear** if every planar curve is similar to a planar curve over some tuple \bar{a} of dimension ≤ 1 .

8.3.4 A lemma.

PROPOSITION Let T be linear (and recall that in this section we always assume $\text{acl } \emptyset$ infinite). Then every planar curve over A is similar to a planar curve that is over some tuple $\bar{a} \subseteq \text{acl } A$ of dimension ≤ 1 .

PROOF Let $\psi(xy)$ be a planar curve over A that is similar to $\varphi(\bar{a}xy)$ for some \bar{a} of dimension ≤ 1 . Say, $\psi(xy)$ and $\varphi(\bar{a}xy)$ coincide up to n points. Let $\vartheta(\bar{z})$ be the formula witnessing that \bar{a} has dimension ≤ 1 . The first-order A -formula saying $\vartheta(\bar{z})$ and that $\psi(xy)$ and $\varphi(\bar{z}xy)$ coincide up to n points is consistent. So it has a solution $\bar{a}' \subseteq \text{acl } A$ (which is a model by the assumption that $\text{acl } \emptyset$ is infinite).

8.3.5 Locally linear \leftrightarrow linear.

Localizing the notion of linearity to a set of parameter E does not yield anything new.

PROPOSITION Let E be set of parameters. Suppose that every planar curve is similar to a planar curve over $E\bar{a}$ for some \bar{a} such that $\dim_E \bar{a} \leq 1$. Then T be linear.

PROOF Let $\psi(\bar{c}xy)$ be a planar curve, where \bar{c} are all the parameters occurring in the formula. Fix some $\bar{b} \equiv \bar{c}$ such that $\bar{b} \downarrow_{\emptyset} E$. Let $\varphi(\bar{e}\bar{a}xy)$ be a planar curve similar to $\psi(\bar{b}xy)$, say equal up to n points, for some $\bar{e} \subseteq E$ and some \bar{a} such that $\dim_{\bar{e}} \bar{a} \leq 1$. There is a first-order formula $\vartheta(\bar{e}\bar{b})$ saying that $\varphi(\bar{e}\bar{z}xy)$ is equal to $\psi(\bar{b}xy)$ up to n points for some \bar{z} such that $\dim_{\bar{e}} \bar{z} \leq 1$. By Proposition 8.2.3 above, there is $\bar{e}' \subseteq \text{acl } \emptyset$ such that $\vartheta(\bar{e}'\bar{b})$. Let \bar{a}' witness $\exists \bar{z}$ in $\vartheta(\bar{e}'\bar{b})$. Then $\varphi(\bar{e}'\bar{a}'xy)$ is a formula over \bar{a}' , where $\dim \bar{a}' \leq 1$, and is similar to $\psi(\bar{b}xy)$. Finally, by homogeneity, we map automorphically \bar{c} to \bar{b} . The image \bar{a}'' of \bar{a}' under this automorphism gives the required planar curve $\varphi(\bar{e}'\bar{a}''xy)$.

8.3.6 Modularity.

We say that **T is modular** if for every algebraically closed sets A and B

$$\# \quad \dim AB = \dim A + \dim B - \dim(A \cap B)$$

This can be written

$$\#\# \quad \dim_A B = \dim_{A \cap B} B.$$

We say that **T is locally modular** if there is a set E such that T is modular when localized to E .

8.3.7 A lemma.

Via pure combinatorial reasoning, we obtain a useful characterization of modularity.

PROPOSITION Let T be strongly minimal. The following are equivalent:

- 1 T is modular;
- 2 whenever $c \in \text{acl}(Ab)$ there is a $a \in \text{acl } A$ such that $c \in \text{acl}(ab)$; and
- 3 whenever $c \in \text{acl}(AB)$ there is a $a \in \text{acl } A$ and $b \in \text{acl } B$ such that $c \in \text{acl}(ab)$.

PROOF For the whole proof we assume A is algebraically closed. We prove 1 \Rightarrow 2 first. Suppose $c \in \text{acl}(Ab)$. Then $\dim(Ab) = \dim(Abc)$. If T is modular, from (#) we obtain

$$\dim A + \dim b - \dim(A \cap \text{acl} b) = \dim A + \dim(bc) - \dim(A \cap \text{acl}(bc))$$

Assume $c \notin \text{acl} b$, otherwise we are done. Then $\dim(bc) = \dim b + 1$ so

$$\dim(A \cap \text{acl} b) = \dim(A \cap \text{acl}(bc)) - 1$$

Hence $A \cap \text{acl}(bc)$ contains an element $a \notin \text{acl} b$. By exchange, from $a \in \text{acl}(bc)$ we obtain $c \in \text{acl}(ab)$.

Now we prove $2 \Rightarrow 3$ by induction on the dimension of B . If B has dimension 0, the claim is clear. So assume the claim holds for B and prove it for Bb where $b \notin \text{acl} B$. If $c \in \text{acl}(ABb)$ then, by (2), there is a $d \in AB$ such that $c \in \text{acl}(db)$. By induction hypothesis $d \in \text{acl}(ae)$ for $a \in A$ and $e \in \text{acl} B$. So $c \in \text{acl}(aeb)$. Apply (2) a second time to find $f \in \text{acl}(eb)$ such that $c \in \text{acl}(af)$. Since $f \in \text{acl}(Bb)$, we are done.

Finally we prove $3 \Rightarrow 1$ so assume (3). We prove (#) by induction on the dimension of B . If B has dimension 0 there is nothing to prove. So suppose the claim holds for some algebraically closed set B and let $c \notin B$. We to prove that

$$\dim(ABc) = \dim A + \dim(Bc) - \dim(A \cap \text{acl}(Bc))$$

We consider two cases: $c \in \text{acl}(AB)$ and $c \notin \text{acl}(AB)$. In the first case we need to show that $A \cap \text{acl}(Bc)$ contains an element not in $A \cap B$. From $c \in \text{acl}(AB)$ and the induction hypothesis we get some $a \in A$ and some $b \in B$ such that $c \in \text{acl}(ab)$. To avoid triviality we assume $b \notin \text{acl} \emptyset$ so, by exchange, we get $a \in \text{acl}(bc)$. So $a \notin A \cap B$, otherwise $c \in B$. In the second case it suffices to show that $A \cap \text{acl}(Bc) = A \cap B$. So observe that if $a \in \text{acl}(Bc) \setminus \text{acl} B$ then, by exchange, $c \in \text{acl}(Ba)$. So $a \notin A$.

8.3.8 Locally modular \leftrightarrow linear.

PROPOSITION Let T be strongly minimal. The following are equivalent:

- 1 T is linear;
- 2 for any $e \notin \text{acl} \emptyset$ the localization of T at e is modular; and
- 3 T is locally modular.

PROOF We prove $1 \Rightarrow 2$ first. Fix any $e \notin \text{acl} \emptyset$ and suppose $c \in \text{acl}_e(Ab)$. We show that $c \in \text{acl}_e(ab)$ for some $a \in \text{acl}_e A$. Assume also that $c \notin \text{acl}_e b$ and $c \notin \text{acl}_e A$, otherwise we are done. So $\dim_{Ae}(bc) = 1$. As $\text{acl}_e A$ is a model, by 8.3.2 above, there is a planar curve over $\text{acl}_e A$ through bc . By linearity this is of the form $\varphi(\bar{a}xy)$ where $\varphi(\bar{z}xy)$ is a parameter-free formula and $\bar{a} \subseteq \text{acl}_e A$ has dimension ≤ 1 . Since we assumed that $c \notin \text{acl} b$, we have $\dim \bar{a} = 1$.

We show that $\bar{a} \in \text{acl}(cb)$. As $\dim \bar{a} = 1$, there is an $a' \notin \text{acl} \emptyset$ such that $\bar{a} \in \text{acl} a'$. Since $c \in \text{acl}(\bar{a}b)$, then $c \in \text{acl}(a'b)$. By symmetry, $a' \in \text{acl}(cb)$. So $\bar{a} \in \text{acl}(cb)$ as desired.

We now prove that $e \notin \text{acl} \bar{a}$ by showing that $a' \notin \text{acl} e$. As cb belong to a planar curve over \bar{a} , then $c \in \text{acl}_{\bar{a}} b$. So, from $a' \in \text{acl} e$ we would obtain $c \in \text{acl}_e b$ which is contrary to the assumptions.

We also have that $c \notin \text{acl} \bar{a}$, so there is a automorphism mapping c to e over \bar{a} . Let a be the image of b under this automorphism. We claim that this a is the required element of $\text{acl}_e A$. Observe that since $b \in \text{acl}(c\bar{a})$, then $a \in \text{acl}_e \bar{a} \subseteq \text{acl}_e A$. So we only have to check that $c \in \text{acl}_e(ab)$. This follows from symmetry if we can show that $a \in \text{acl}_e(cb)$ and $a \notin \text{acl}_e b$. The first is clear since $\text{acl}_e \bar{a} \subseteq \text{acl}_e(cb)$. For the second observe that $b \notin \text{acl}_e a$ and that $a \in \text{acl}_e \emptyset$, so $a \notin \text{acl}_e b$ follows by symmetry.

The implication $2 \Rightarrow 3$ does not need a proof so only the implication $3 \Rightarrow 1$ is left. The argument is clearer if we assume that T is modular. Let $\varphi(\bar{a}xy)$ be a planar curve and let cb be a pair in the curve not in $\text{acl} \bar{a}$. Replacing A with $\text{acl} \bar{a}$ and $\text{acl}(cb)$ for B in (###) of 8.3.6 we obtain that the dimension of cb over $\text{acl}(cb) \cap \text{acl} \bar{a}$ is 1. Since $\text{acl}(cb) \cap \text{acl} \bar{a}$ has dimension 1 and is a model, the claim follows from Proposition 8.3.2 above. In general, i.e. when T is simply locally modular, we need just to fix a set E containing the the parameters that make T modular and in any case infinite. Then the argument above yields that T is linear over E so, by 8.3.5 above, linear.

Chapter 9

Prime and atomic models

It would be very convenient to have a general notion of model generated by a set A of parameters. The algebraic closure of A would be the natural candidate but, in general, it is not a model. More possibilities are offered if *contained in* every model containing A is replaced with *embeddable over* A in every model containing A . In this way we naturally come to the notion of model *prime over* A . In this chapter we also introduce the notion of model *atomic over* A in fact this is deeply (often non-trivially) connected to the notion of prime model. We shall also meet strictly-prime models: a technical tools for the construction of prime and atomic models.

Notation and global assumptions.

As in Chapter 6 above.

9.1 Prime and atomic models

In this section we define prime models. Unfortunately, there is no syntactic notion that obviously corresponds to primality. We shall define the notion of atomic model that works well when A is countable (as we see in 10.1.1 below) and, to cope with the uncountable case, we introduce also the notion of strictly-prime model.

9.1.1 Prime models.

Let $A \subseteq M$. We say that **M is prime over A** if it is A -elementary embeddable in every model that contains A . In other words, whenever $A \subseteq N$ there is an A -automorphism of the universe which takes M into a subset of N . When A is empty we simply say that M is prime.

9.1.2 Examples and non-examples.

Let T be the theory of dense linear orders without endpoints. The reader can prove as an exercise that a prime model exist over any set A . EXERCISE

9.1.3 Isolation.

Recall that a formula $\varphi(\bar{x})$ is **complete over B** , if for every B -formula $\psi(\bar{x})$ exactly one of $\varphi(\bar{x}) \wedge \psi(\bar{x})$ and $\varphi(\bar{x}) \wedge \neg\psi(\bar{x})$ is consistent. Equivalently, if $\varphi(\bar{x})$ is consistent and either $\varphi(\bar{x}) \rightarrow \psi(\bar{x})$ or $\varphi(\bar{x}) \rightarrow \neg\psi(\bar{x})$ holds. When B is clear from the context we just say complete. E.g. when φ has been introduced as an A -formula, then *complete* is, by default, complete over A . We say that **$\varphi(\bar{x})$ isolates \bar{c} over B** if $\varphi(\bar{c})$ holds and $\varphi(\bar{x})$ is complete over B . Again when φ has been introduced as a A -formula then A is the default value of B . We say that **A isolates \bar{c} over B** if some A -formula isolates \bar{c} over B . The notion of isolation is often used when $A = B$; in this case we say either that **A isolates \bar{c}** or that **\bar{c} is isolated over A** .

9.1.4 Isolating elements versus isolating types.

Roughly speaking, isolated tuples have isolated types (these have been defined in 7.3.2 above) and vice versa. In fact, let $p(\bar{x})$ be an A -type and let \bar{c} be a realization of $p(\bar{x})$. If $\varphi(\bar{x})$ isolates \bar{c} over A then $\varphi(\bar{x})$ isolates $p(\bar{x})$. When $\varphi(\bar{x})$ is an A -formula and $p(\bar{x})$ is complete then the converse holds as well, in fact, any A -formula that isolates a complete A -type is satisfied by all realizations of the type. But one has to bear in mind that this holds only when the parameters of $p(\bar{x})$ and $\varphi(\bar{x})$ match. For instance for every realization \bar{c} of $p(\bar{x})$ the formula $\bar{c} = \bar{x}$ isolates $p(\bar{x})$ but it does not isolate any other realization of $p(\bar{x})$.

9.1.5 Atomic models.

Observe that an A -formula $\varphi(\bar{x})$ is complete (over A) if the set $\varphi(\bar{\mathcal{U}})$ is an atom of $\text{Def}_A \mathcal{U}$, the Boolean algebra of the A -definable sets (by abuse of language here we think of $\text{Def}_A \mathcal{U}$ as the set of definable sets of the same fixed arity; in this case the arity of \bar{x}). We say that the model **M is atomic over A** if $A \subseteq M$ and A isolates (over A) every tuple $\bar{a} \subseteq M$. In other words, if every tuple of elements of M belongs to an atom of the Boolean algebra $\text{Def}_A \mathcal{U}$.

9.1.6 A warning.

Unfortunately the notion of isolation used in the definition of atomicity may depend very wildly on A . Completeness over B is obviously preserved if we restrict B while the property of being an A -formula is obviously preserved if we enlarge A . But when

we require $A = B$, things get messed up. Consequently the notion of atomicity is sensitive to any change of parameters.

9.1.7 Prime and atomic: trivial examples.

Prime and atomic models need not exist in general. It is easy to find a simple and natural condition on T which enforces the existence of models which are both prime and atomic, namely: every consistent formula has a solution which is algebraic over the parameters of the formula; in Exercise 7.1.12 we called theories with this property: theories with algebraic witnesses. If T has algebraic witnesses, then $\text{acl}(A)$ is a model for every A . This model is prime in a very strong (trivial) sense: $\text{acl}(A)$ is actually included in every model containing A . Since algebraic elements are isolated (cf. Exercise 7.3.5 above), then $\text{acl}(A)$ is also atomic over A .

9.1.8 Atomic theories.

Theories like those considered in the paragraph above (i.e. theories with algebraic witnesses) have *built-in* a way of naming (finite sets of) witness of consistent formulas. This property trivializes all questions about existence and uniqueness of prime and atomic models. There are more natural conditions that guarantees the existence of prime and atomic models.

We say that T is **atomic** if for every consistent parameter-free formula is implied by some complete (over \emptyset) parameter-free formula. In the next chapter we will see that atomic theories are exactly those that have prime models over \emptyset . This is in general insufficient to guarantee the existence of atomic models over arbitrary sets.

In this chapter we want to analyze the relations between atomic, prime, strictly prime models over arbitrary sets without bothering about existence problems. So we introduce the following (non standard) notion. We say that T is **atomic over any set of parameters** if every consistent A -formula is implied by some A -formula which is complete over A . In other words, every consistent A -formula has a solution which is isolated over A . (This notion seems to depend on the universe, a short reflection shows that it is really a property of T . We will see that ω -stable theories are atomic over any set of parameters.)

9.1.9 Strictly-prime models.

There is a natural strategy to construct prime and atomic models. This inspires the introduction of a technical concept which will prove very useful below. We say that M is **strictly-prime over A** if M is the union of a chain of length $|M|$ which starts with $A_0 = A$ and is such that $A_{\alpha+1} = A_\alpha + a_\alpha$ for some a_α isolated over A_α . This chain is called a **construction sequence of M over A** .

9.1.10 Atomic theories: strictly-prime models exist.

PROPOSITION Let T be atomic over any set of parameters. For every set A there is a model M which is strictly-prime over A .

PROOF We will exhibit a construction sequence over A of the model M . Let $A_0 = A$ and, at stage α , consider the least consistent formula $\varphi(x)$ which has parameters in A_α but is not satisfied there. Let a_α be any witness of $\varphi(x)$ which is isolated over A_α . This witness exist by the atomicity of T .

9.1.11 A lemma on isolation.

We want to prove that $\bar{a}\bar{b}$ is isolated over A if and only if \bar{a} is isolated over A and \bar{b} is isolated over $A + \bar{a}$. One direction is consequence of next proposition (the claim is somewhat more precise), the converse is proved in 9.1.13 below.

PROPOSITION If $\varphi(\bar{x})$ isolates \bar{a} over A then:

- 1 if A isolates \bar{b} over $A + \bar{a}$ then $\varphi(\bar{x})$ isolates \bar{a} over $A + \bar{b}$; and
- 2 if $\psi(\bar{a}\bar{y})$ isolates \bar{b} over $A + \bar{a}$ then the formula $\varphi(\bar{x}) \wedge \psi(\bar{x}\bar{y})$ isolates $\bar{a}\bar{b}$ over A .

PROOF To prove 1 assume the hypothesis and let $\xi(\bar{a}\bar{b})$ be some true A -formula. Let $\psi(\bar{y})$ be an A -formula which isolates \bar{b} over $A + \bar{a}$. So, $\psi(\bar{y}) \rightarrow \xi(\bar{a}\bar{y})$ holds. Now use that $\varphi(\bar{x})$ isolates \bar{a} over A to obtain $\varphi(\bar{x}) \wedge \psi(\bar{y}) \rightarrow \xi(\bar{x}\bar{y})$. Substituting \bar{b} for \bar{y} we obtain what is required. The proof of 2 is very similar: assume the hypothesis and let $\xi(\bar{x}\bar{y})$ be some A -formula that is true at $\bar{a}\bar{b}$. Then $\psi(\bar{a}\bar{y}) \rightarrow \xi(\bar{a}\bar{y})$ holds by the completeness of $\psi(\bar{a}\bar{y})$. Now use that $\varphi(\bar{x})$ isolates \bar{a} over A to obtain $\varphi(\bar{x}) \wedge \psi(\bar{x}\bar{y}) \rightarrow \xi(\bar{x}\bar{y})$ which proves the claim.

9.1.12 A remark.

The following is straightforward but it is worth noting. In particular, one should realize that the claim below would no longer be true if the formula $\varphi(\bar{x})$ is replaced with a set A . The following claim can be verified directly.

PROPOSITION A formula $\varphi(\bar{x})$ isolates \bar{a} over B if and only if it isolates \bar{a} over every tuple $\bar{b} \subseteq B$.

9.1.13 Isolated tuples have isolated components.

The following proposition claims that the isolation of a tuple implies the isolation of all its components. It can be verified directly.

PROPOSITION If $\psi(\bar{x}\bar{y})$ isolates $\bar{a}\bar{b}$ over A then:

- 1 $\psi(\bar{a}\bar{y})$ isolates \bar{b} over $A + \bar{a}$; and
- 2 $\exists \bar{x} \psi(\bar{x}\bar{y})$ isolates \bar{b} over A .

9.1.14 A remark.

An immediate consequence of the lemma above is that a model atomic over A is atomic over any finite extension of A .

9.1.15 Strictly-prime \rightarrow atomic.

The definition of strictly-prime models looks much like that of atomic models. But some non-trivial use of the combinatorics of isolation is necessary to obtain atomicity.

PROPOSITION If M is strictly-prime over A then M is atomic over A .

PROOF We show by induction on α that every tuple $\bar{a} \subseteq A_\alpha$ is isolated over A . For limit ordinals induction is clear, so we just need to prove the claim for $A_{\alpha+1}$ assuming it is true for A_α . Consider a tuple of the form $\bar{a} a_\alpha$ where \bar{a} is in A_α . This is general enough, in fact: for tuples in $A_{\alpha+1}$ which do not contain a_α there is nothing to prove and any other tuple has, up to a permutation, the $\bar{a} a_\alpha$. By construction A_α isolates a_α over A_α . Fix a tuple $\bar{c} \subseteq A_\alpha$ such that $A + \bar{a} \bar{c}$ isolates a_α over A_α . By the induction hypothesis, A isolates $\bar{c} \bar{a}$ over A . From Lemma 9.1.11 it follows that A isolates $\bar{c} \bar{a} a_\alpha$ over A (the second claim with $\bar{a} \bar{c}$ for \bar{a} and a_α for \bar{b}). Finally, from Lemma 9.1.13 we obtain the isolation of $\bar{a} a_\alpha$ over A .

9.1.16 Strictly-prime \rightarrow prime.

Strictly-prime models are designed to be easily embeddable in every model: the construction sequence tells explicitly the order in which elements have to be embedded.

PROPOSITION If M is strictly-prime over A then M is prime over A .

PROOF Let N be any model containing A . We construct an embedding $F : M \rightarrow N$ in stages. Let $A_{\alpha+1} = A_\alpha + a_\alpha$ be a construction sequence of M over A . At stage α we define the A -elementary map F_α with domain A_α . We begin with $F_0 = \text{id}_A$ and at limit stage we take the union. Limit stages are unproblematic by the elementary chain lemma for maps. So consider stage $\alpha + 1$. By hypothesis there exist $\bar{a} \in A_\alpha$ and a parameter-free formula $\psi(x \bar{a})$ that isolates a_α over A_α . By the induction hypothesis we have that $\psi(x F_\alpha \bar{a})$ is a complete formula over $\text{rng } F_\alpha$, in fact, the property of being a complete formula is preserved by elementary maps. Now, let c_α be any arbitrary element of N such that $\psi(c_\alpha F_\alpha \bar{a})$. By the completeness of $\psi(x \bar{a})$, the map $F_{\alpha+1} = F_\alpha + \langle a_\alpha, c_\alpha \rangle$ is an A -elementary map.

9.1.17 The theory is atomic \leftrightarrow atomic models exist.

To sum up, Propositions 9.1.10 and 9.1.15 above shows if T is atomic over any set of parameters then for every A , there is a model atomic over A (the converse holds obviously).

9.1.18 Atomic theories: prime \rightarrow atomic.

The following is an easy observation; its converse is more interesting since it holds for countable theories (cf. Proposition 10.1.1 below) but fails in general. In some cases this failure may be related to the presence of indiscernibles in the model: we shall not deal with this topic here; the interested reader may consult [BUECHLER], [LASCAR] or [PILLAY I] (knowledge of the notions and the results of Chapters ?? and ?? is required).

PROPOSITION Let T be atomic over any set of parameters. If M is prime over A then M is atomic over A .

PROOF There is a strictly-prime model N over A . Since N is atomic over A , every submodel of N containing A is also atomic. Since M is prime, it embeds in N over A , so it is isomorphic to one of these submodels. It follows that M is atomic over A . Let

9.1.19 A theory without prime models

We give an example of a countable theory that has no prime models over \emptyset . Let T be the theory introduced in 4.3.8 above. Recall that by 5.3.9 above, T is a complete theory with quantifier elimination. Let M be a model of T . The set of $s \in {}^\omega 2$ such that the type

$$p_s(x) := \{r_i(x) : s(i) = 1\} + \{\neg r_i(x) : s(i) = 0\}$$

is realized in M is a dense subset of ${}^\omega 2$. Vice versa, for every dense subset $\mathcal{X} \subseteq {}^\omega 2$ there is an model N of T that realizes the types $p_s(x)$ iff $s \in \mathcal{X}$. It follows that for every model M we can find a model M' omitting a type that is realized in M . there is no model that embeds in all the others.

9.1.20 An exercise.

We say that a model is **minimal** if it has no proper submodel (do not confuse this with the homonymous notion defined in 8.1.4 below). Let T be a countable theory. Suppose T has a prime model M which is not minimal. Prove that there is an atomic model of T of cardinality \aleph_1 . EXERCISE

9.1.21 An exercise.

Let T be a countable theory. Suppose T is uncountably categorical but not totally categorical. Let M be a prime model of T . Prove that M is minimal. EXERCISE

9.1.22 An exercise.

Let T be a complete theory (there is no assumption on the cardinality of T). Let M be a prime model of T and let N be a minimal model of T . Prove that M and N are isomorphic. EXERCISE

9.2 Uniqueness of strictly-prime models

While countable prime models are unique up to isomorphism (this we see in the next chapter) uncountable prime models may not be unique. For a counter-example the reader may consult [POIZAT]. Here we prove the uniqueness up to isomorphism of strictly-prime models.

9.2.1 Families of closed sets.

We introduce some technical notions that will help us to expose the main result of this section. Let M be an arbitrary model. We say that a non-empty family of subsets of M is a **family of closed subsets of M** if

- 1 for every closed set B and every tuple $\bar{a} \subseteq M$ there is a tuple $\bar{c} \subseteq M$ such that $B + \bar{a}\bar{c}$ is again closed: we say that the family is **dense**; and
- 2 the union of a chain of length $< |M|$ of finite extensions of closed set is closed: we say that the family is **inductive**.

9.2.2 Strictly-atomic models.

The following definition is non-standard. We say that **M is strictly-atomic over A** if there is a family of closed sets such that A is closed and M is atomic over all closed sets.

9.2.3 Uniqueness of strictly-atomic models.

Strictly-prime models have been designed with the proof of existence in mind, strictly-atomic models instead have been defined to be applied in the following proof.

PROPOSITION Let M and N be two models strictly-atomic over A and of the same cardinality. Then M and N are isomorphic over A .

PROOF Fix a family of closed subsets of M as required by the definition of strictly-atomic model and do the same for N . We construct an A -isomorphism $G : M \rightarrow N$ as union of a chain $G_\alpha : M \rightarrow N$ of elementary maps. We require that $G_{\alpha+1}$ is a finite extension of G_α . We also require that: **#** if α is even the range of G_α is closed (in N) and if α odd the support of G_α is closed (in M). Observe that **(#)** implies, by inductivity, that at limit stages both the domain of definition and the

range of F are closed. So **(#)** is preserved at limit stages. The chain begins with the identity on A , so at stage 0 both support and range are closed. Suppose α is even (the odd case is similar and it is left to the reader). The support of G_α is the finite extension of a closed set. So there is a tuple \bar{a} such that $\text{dom } G_\alpha + \bar{a}$ is closed. We can further assume that \bar{a} contains the least element of M not yet in the support of G_α so that in the limit we obtain a total map. By the remark in 9.1.14 above, M is atomic over the support of G_α , so there is a formula $\varphi(\bar{x})$ over $\text{dom } G_\alpha$ that isolates \bar{a} . Let \bar{b} be a tuple of N satisfying $G_\alpha\varphi(\bar{x})$ and define $G_{\alpha+1}$ to be $G_\alpha + \langle \bar{a}, \bar{b} \rangle$. It is easy to see that this is an elementary map (because $\varphi(\bar{x})$ is complete over the domain of definition of G_α).

9.2.4 Closed subsets in a strictly-prime model.

We are close to prove that a model is strictly-atomic if and only if it is strictly-prime. Here, given a construction sequence of M over A , we define a family of closed subsets of M . Let $A_{\alpha+1} = A_\alpha + a_\alpha$ be a construction sequence of M over A . We say that B is a **closed subset of M** if $A \subseteq B \subseteq M$ and for every $a_\alpha \in B$ we have that $B \cap A_\alpha$ isolates a_α over A_α . That is, the parameters necessary to isolate the elements of B can be found in B itself. (When a construction sequence is fixed in the context *closed* always refers to family of closed sets associated to the construction sequence.) We need to check density and inductivity. Inductivity is clear from the definition. So we need only prove: ***** for every \bar{a} there is a \bar{c} such that $B + \bar{a}\bar{c}$ is closed. To this end, observe that ****** if B is closed and $B \cap A_\beta$ isolates a_β over A_β then $B + a_\beta$ is closed as well. So prove density we proceed by induction: suppose **(*)** is true for every tuple $\bar{a} \subseteq A_\beta$ and prove it is true for every tuple $\bar{a} a_\beta$ with $\bar{a} \subseteq A_\beta$. Let $\varphi(x)$ be a A_β -formula that isolates a_β over A_β . By induction hypothesis there is a \bar{c} containing the parameters of $\varphi(x)$ and such that $B + \bar{a}\bar{c}$ is closed. By observation **(**)** above $B + \bar{a} a_\beta \bar{c}$ is closed.

9.2.5 An exercise.

We leave the proof of the following fact to the reader as an exercise.

PROPOSITION Let M be strictly prime over A . Let B be a closed subset of M . Then $A_\alpha \cap B$ isolates \bar{b} over A_α , for every $\bar{b} \subseteq B$. EXERCISE

9.2.6 Strictly-prime \leftrightarrow strictly-atomic.

PROPOSITION The following are equivalent:

- 1 M is strictly-prime over A ; and
- 2 M is strictly-atomic over A .

The proof of $2 \Rightarrow 1$ is left to the reader as an exercise. To prove $1 \Rightarrow 2$ we fix a construction sequence of M over A . From this construction sequence we obtain a family of closed sets as explained above. It suffices to show that M is atomic over every closed set B . We claim that M is strictly-prime over B via the construction sequence BA_α . To prove claim we check every $a_\alpha \notin B$ is isolated over BA_α by the same formula $\varphi(x)$ which isolates a_α over A_α . By 9.1.12 above, it suffices to prove that $\varphi(x)$ isolates a_α over $A_\alpha + \bar{b}$ for every $\bar{b} \subseteq B$. So fix $\bar{b} \subseteq B$. By the first claim of Proposition 9.1.11 above, it is enough to show that A_α isolates \bar{b} is isolated over $A_\alpha + a_\alpha$. We know that $A_\alpha + a_\alpha$ isolates \bar{b} . But B is closed so, by the exercise above, $(A_\alpha + a_\alpha) \cap B$ isolates \bar{b} over $A_\alpha + a_\alpha$. Since $a_\alpha \notin B$ we conclude that $A_\alpha \cap B$ isolates \bar{b} over $A_\alpha + a_\alpha$. EXERCISE

Chapter 10

Countable models

The omitting types theorem allows to sharpen the theory of prime and atomic models. This will be used to prove some fundamental results about countable categoricity. Finally we apply the technologies developed so far to prove a few nice results: no complete theory has exactly two non-isomorphic models; and ω -categorical strongly minimal theories are not finitely axiomatizable.

Notation and global assumptions.

As in Chapter 6 above.

10.1 Countable atomic models

Countable prime and atomic models are easier to construct and simpler to understand. For instance, the three notions introduced in the previous chapter collapse to one in the case of countable models.

10.1.1 Countable models: atomic \leftrightarrow prime \leftrightarrow strictly-prime.

PROPOSITION Assume that L is countable. For every countable set A and every model M , the following are equivalent:

- 1 M is prime over A ;
- 2 M is countable and atomic over A ; and
- 3 M is strictly-prime over A .

PROOF The implication $1 \Rightarrow 2$ is an immediate corollary of the omitting types theorem: a model that realizes a non-isolated A -type cannot be embedded in a model which omits this type. Since a countable model containing A exists, then prime models have to be countable. To prove that $2 \Rightarrow 3$ fix an arbitrary enumeration a_i for $i \in \omega$ of M . We claim that the chain $A_0 = A$ and $A_{i+1} = A_i + a_i$ is a construc-

tion sequence of M over A . In fact, for i arbitrary let $\varphi(x_0 \cdots x_i)$ be an A -formula which isolates $a_0 \cdots a_i$ over A . By Proposition 9.1.13 above $\varphi(a_0 \cdots a_{i-1} x_i)$ isolates a_i over $A + a_0 \cdots a_{i-1}$. This proves the claim. The implication $3 \Rightarrow 1$ has been proved in 9.1.15 above, so the proof is complete.

10.1.2 Uniqueness of countable prime models.

In the countable case prime models coincide with strictly-prime models so the following uniqueness result follows directly from the uniqueness of strictly-prime models which we proved in Section 9.2 above. But, since the proof in the countable case is much easier, we give also a direct proof.

PROPOSITION Let L and A be countable. Suppose that M and N are both prime over A . Then M and N are isomorphic over A .

PROOF Clearly M and N are countable. Fix some well-orderings of length ω of M and N . We shall construct in ω stages an A -isomorphism $F : M \rightarrow N$ as limit of a chain $F_i : M \rightarrow N$ of finite A -elementary maps. The chain has length ω and starts with the empty map. At stage $i + 1$ assume as induction hypothesis that a finite A -elementary map $F_i : M \rightarrow N$ has been defined; we will define F_{i+1} . Assume first that i is even. Let \bar{a} be a tuple containing all the elements of $\text{dom } F_i$ and let a_i be the least element of M not in $\text{dom } F_i$. Let $\varphi(y \bar{x})$ be an A -formula which isolates $a_i \bar{a}$ over A . Then the formula $\varphi(y \bar{a})$ isolates a_i over $A + \bar{a}$. By induction hypothesis, the formula $\varphi(y F_i \bar{a})$ is consistent and complete over $A + F_i \bar{a}$. Let c_i be an element of N that satisfies $\varphi(y F_i \bar{a})$. Let F_{i+1} be $F_i + \langle a_i, c_i \rangle$. When i odd, we use the same procedure but reverse the role of domain and codomain.

10.1.3 Atomic models over finitely many parameters.

In this chapter we work with the parameter-free version of atomicity and isolation. Some existence results of Chapter 9 are easily adapted to this context.

PROPOSITION Let L be countable. The following are equivalent:

- 1 T is atomic;
- 2 for every A finite, every A -formula consequence of some complete A -formula; and
- 3 there is an atomic model.

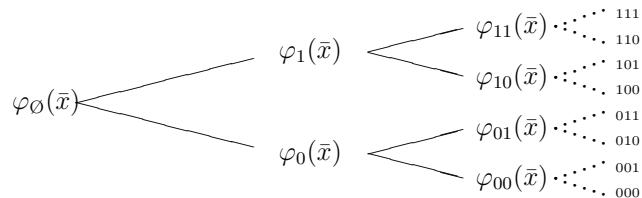
PROOF The implication $1 \Rightarrow 2$ is easy and can be entrusted to the reader. To prove the implication $2 \Rightarrow 3$ we construct a strictly prime model over \emptyset using the proof in 9.1.10 above: observe that we only need 2 to carry on this construction for ω stages. The implication $3 \Rightarrow 1$ is trivial.

10.1.4 Existence of countable prime models.

There is an easy to check property that guarantees the atomicity of T .

PROPOSITION If there are less than 2^{\aleph_0} complete parameter-free types, then T is atomic.

PROOF We prove that every parameter-free formula is implied by some complete parameter-free formula. So suppose for a contradiction that there is a parameter-free formula $\varphi(\bar{x})$ which is not implied by any complete parameter-free formula. We shall construct an infinite binary tree of formulas with root $\varphi(\bar{x})$. To each $s \in {}^{<\omega}2$ (the set of finite sequences of 0's and 1's) we assign a formula $\varphi_s(\bar{x})$. We require that every branch of the tree depicted below is consistent but that different branches are pairwise inconsistent.



Explicitly, we require the consistency, for every $s \in {}^{<\omega}2$, of the formulas $\psi_s(\bar{x})$ that are defined to be the conjunction of $\varphi_r(\bar{x})$ for $r \subseteq s$ and we require the inconsistency of $\varphi_{s0}(\bar{x}) \wedge \varphi_{s1}(\bar{x})$. Clearly the existence of the tree above implies immediately the existence of 2^{\aleph_0} many parameter-free types. The construction of the tree begins with setting for φ_\emptyset , the root of the tree, the formula $\varphi(\bar{x})$. Then we assume that $\psi_s(\bar{x})$ has been defined and proceed by induction. Since $\psi_s(\bar{x})$ implies $\varphi(\bar{x})$ then, by the assumption above, $\psi_s(\bar{x})$ is not complete. So there exists $\xi(\bar{x})$ such that both $\psi_s(\bar{x}) \wedge \xi(\bar{x})$ and $\psi_s(\bar{x}) \wedge \neg\xi(\bar{x})$ are consistent: let $\xi(\bar{x})$ be $\varphi_{s0}(\bar{x})$ and let $\varphi_{s1}(\bar{x})$ be $\neg\xi(\bar{x})$.

10.1.5 An exercise.

Let L be countable. Prove that there are either finitely, countably, or continuum many parameter-free types. **EXERCISE**

10.1.6 Countable prime models are homogeneous.

Recall that a model M is homogeneous over A if every A -elementary map $F : M \rightarrow M$ of cardinality $< |M|$ extends to an automorphism of M . The reader can prove the following proposition as an exercise. Note that the uniqueness of countable atomic models is a consequence of the following proposition. In fact, the theory developed in Section 6.2 generalizes to A -homogeneity; in particular we have that two A -homogeneous models of the same cardinality are A -isomorphic if and only

if they realize the same A -types. Since atomic models realize just isolated types, uniqueness follows.

PROPOSITION Let M be a countable model and let $A \subseteq M$. If M is atomic over A than M is homogeneous over A . **EXERCISE**

10.2 Countable categoricity

We prove some important classical theorems that characterize ω -categorical theories in different interesting ways.

10.2.1 Countable categoricity.

We recall the definition of ω -categoricity. We say that T is **ω -categorical** if any two model are isomorphic. We met some examples of ω - theories in Section 11.4 above: dense linear orders without endpoints, the theory of the countable random graph. For the time being we find convenient to introduce the following terminology: we say that T is *ω -categorical over A* if any two model containing A are isomorphic over A . At the end of this section we see that either A is infinite, then this notion is empty, or A is finite then this notion is equivalent to ω -categoricity.

10.2.2 The Engeler, Ryll-Nardzewsky, and Svenonius theorem.

This is the famous characterization of ω -categoricity mentioned above. The second property below can be stated in different equivalent ways; for convenience, these are considered in a separate lemma.

PROPOSITION Let L be countable. Let A be a countable set. The following are equivalent:

- 1 T is ω -categorical over A ; and
- 2 every A -type is isolated (by an A -formula).

PROOF The implication 1 \Rightarrow 2 is an immediate consequence of the omitting types theorem. If $p(\bar{x})$ is a non-isolated A -type, then there are two countable models M and N containing A such that M realizes $p(\bar{x})$ and N omits $p(\bar{x})$. Clearly M and N cannot be A -isomorphic. To prove the converse implication 2 \Rightarrow 1 observe that every countable model containing A is atomic over A . But countable atomic models are prime and, by 10.1.2 above, unique up to isomorphism.

10.2.3 Other equivalents to countable categoricity.

Requiring that all types are isolated has a dramatic impact on the structure of definable sets. Recall that $S_n(A)$ is the set of complete A -types $p(\bar{x})$ where \bar{x} is

some fixed tuple of variables of arity n .

PROPOSITION Fix a set A and a tuple of variables \bar{x} of arity n . The following are equivalent:

- 1 every A -type $p(\bar{x})$ is isolated;
- 2 $S_n(A)$ is finite;
- 3 there is finite number of A -formulas $\varphi_1(\bar{x}) \cdots \varphi_m(\bar{x})$ such that any other A -formula $\varphi(\bar{x})$ is equivalent to one of these;
- 4 there is a finite number of orbits in \mathcal{U}^n under $\text{Aut}_A \mathcal{U}$.

PROOF To prove the implication $1 \Rightarrow 2$ observe that \bar{U} is the union of sets of the form $p(\bar{u})$ where $p(\bar{x})$ is a complete A -type. If these types are isolated then \mathcal{U} is the union of A -definable sets. By saturation this union has to be finite. To prove the converse implication $2 \Rightarrow 1$ let $p(\bar{x})$ be a complete A -type. If $S_n(A)$ is finite, the set $\bar{U} \setminus p(\bar{U})$ is the union of finitely many A -type definable sets. Finite union of type definable sets are type definable. So $\bar{U} \setminus p(\bar{U})$ is type definable. Hence $p(\bar{U})$ is isolated. We prove the implication $2 \Rightarrow 1$ observe that every A -formula is equivalent to the disjunction of the complete A -types that contain this formula. If $S_n(A)$ is finite, finitely many disjunctions are possible. The implication $3 \Rightarrow 2$ is clear and the equivalence $2 \Leftrightarrow 4$ follows from the characterization of orbits as type-definable sets, see 6.2.11 above.

10.2.4 Adding parameters.

If any of the four facts above holds for some set A then it holds for $A + \bar{a}$ for any tuple \bar{a} and vice versa (using the third of the equivalents above this claim is immediate to check). So we have that T is ω -categorical if and only if T is ω -categorical over some finite set A . On the other side, note that the third equivalent above cannot hold if there are infinitely many A -definable elements. So no theory can be ω -categorical over an infinite set A .

10.2.5 Yet another characterization of countable categoricity.

Let L be countable. The reader can prove as an exercise that the theory T is ω -categorical if and only if there is a saturated model that is prime over some finite set. EXERCISE

10.2.6 An exercise.

Let T be ω -categorical and let A be a finite set. Prove that $\text{acl } A$, so in particular the substructure generated by A , is finite. EXERCISE

10.2.7 An exercise.

Prove that no theory can have a model that is both atomic and saturated over an infinite set of parameters. EXERCISE

10.2.8 An exercise.

Let L be a finite and without functions of positive arity. Prove that if T has elimination of quantifiers then it is ω -categorical. EXERCISE

10.3 Finite axiomatizability

As an application of the characterization of ω -categoricity given above we prove the non finite axiomatizability of theories that are both strongly minimal and ω -categorical.

10.3.1 Homogeneous sets

Homogeneous sets have been defined in 6.2.19 above. When C is finite (or countable) this is equivalent to the following: for every $\bar{a}, \bar{b}, c \subseteq C$ such that $\bar{a} \equiv \bar{b}$ there is a $d \in C$ such that $\bar{a}c \equiv \bar{b}d$.

10.3.2 A lemma.

PROPOSITION Suppose that T is ω -categorical and that every finite set is contained in a finite homogeneous set. Then T is not finitely axiomatizable.

PROOF We show that for every true parameter-free sentence φ there is a finite substructure A of the universe such that $A \models \varphi$. This is called *the finite model property*, from it the theorem follows. In fact, since T has no finite models, no true sentence can imply the whole of T . We will prove that for every n there is a finite structure A such that $A \models \varphi$ for every true A -sentence φ such that

- * number of parameters of φ + number of quantifiers of $\varphi \leq n$.

By ω -categoricity, there is a finite set A containing a solution of each consistent parameter-free formulas with n free variables. By hypothesis we can assume that A is a finite homogeneous set. The proof that A is a structure can be omitted: it is identical to the reasoning below with $f(\bar{b}) = x$ for $\varphi(\bar{b}x)$. To prove that A models every true sentence φ such that (*), we proceed by induction on the complexity of φ . If φ is atomic there is nothing to prove and induction for the Boolean quantifiers is obvious. So let φ have the form $\exists x \varphi(\bar{b}x)$ for some $\bar{b} \subseteq A$ and $\varphi(\bar{z}x)$ parameter-free. Since the arity of $\bar{z}x$ does not exceed n and $\varphi(\bar{z}x)$ is consistent then it has a solution in A , let this be $\bar{a}c$. Since by ω -categoricity the parameter-free type of \bar{b}

is isolated we can also require that $\bar{a} \equiv \bar{b}$. Then by homogeneity there is a $d \in A$ such that $\bar{a}c \equiv \bar{b}d$. In particular $\varphi(\bar{b}d)$. By induction hypothesis $A \models \varphi(\bar{b}d)$ so $A \models \exists x \varphi(\bar{b}x)$.

10.3.3 A toy version of Zilber's theorem.

The following result, with *uncountably categorical* (or even *superstable*) substituted for *strongly minimal*, is a major result of Zilber and independently of Cherlin, Harrington and Lachlan: the breakthrough that signed the birth of the so-called geometric model theory (or geometric stability theory). This should not suggest false expectations: the real theorem is some hundred of heavy pages away from this toy version. (The argument used below occurs also in the proof of Zilber's theorem, so it will bring the reader one page ahead.) To my knowledge, the best exposition of Zilber's theorem is to be found in [PILLAY II].

PROPOSITION Let T be strongly minimal and ω -categorical. Then T is not finitely axiomatizable.

PROOF Observe that when T is ω -categorical the algebraic closure of a finite set is finite. So the proposition follows from the corollary above if we can show that every algebraically closed set in a strongly minimal theory is homogeneous. To prove this claim let C be an algebraically closed set and let $\bar{a}, \bar{b}, c \subseteq C$ be such that $\bar{a} \equiv \bar{b}$. Suppose first that $c \in \text{acl}\bar{a}$ then let F be any automorphism of the universe that maps \bar{a} to \bar{b} . Since by Exercise 7.1.17 above, $F[\text{acl}\bar{a}] = \text{acl}\bar{b}$, letting $Fc = d$ we obtain $\bar{a}c \equiv \bar{b}d$. If otherwise $c \in C \setminus \text{acl}\bar{a}$ then since $\bar{a} \equiv \bar{b}$ it follows that also $C \setminus \text{acl}\bar{b}$ is non-empty (otherwise C would have two bases of different cardinality). For every $d \in C \setminus \text{acl}\bar{b}$ we have that $\bar{a}c \equiv \bar{b}d$ in fact: if $\varphi(\bar{a}c)$ holds, then $\varphi(\bar{a}x)$ is non-algebraic, then $\varphi(\bar{b}x)$ is non-algebraic, then $\varphi(\bar{b}d)$ holds, and vice versa.

10.4 On the number of countable models

There are few things one can say in general on the number of non-isomorphic models of countable cardinality. In this section we show that with the only exception of 2 all finite cardinalities are possible.

10.4.1 Complete theories with infinitely many countable models.

It is easy to find natural examples of countable theories with \aleph_0 many countable models e.g. the theory of vector spaces over the rationals has a countable models for every finite dimension, one of infinite dimension. There are also plenty of complete theories with 2^{\aleph_0} many countable models for instance (though this may be not immediately obvious) the theory of the natural numbers in the language $\{0, 1, +, \cdot\}$.

For a more artificial (shorter to work out) example take the theory of example 4.3.8 above: the collection of countable dense subsets of 2^ω gives a continuum of pairwise non-isomorphic models of that theory. Finally, note that it is not known if a countable complete theory exist with λ many countable models where $\aleph_0 < \lambda < 2^{\aleph_0}$. A famous conjecture of Vaught says that there is no such theory. This has been confuted by Robin W. Knight in 2002.

10.4.2 A complete theory with exactly three countable models.

This is a famous example that contrasts with a theorem below. Let L contain just the binary relation $<$. Let T be a theory of dense linear orders without endpoints. Let $A = \{a_i : i \in \omega\}$ be a subset of \mathcal{U} such that $a_i < a_{i+1}$ for every $i \in \omega$. The reader can prove as an exercise that up to A -isomorphism there are exactly three countable models containing A . (Hint. Let M be a model containing A and let $B \subseteq M$ be the set of those elements of M that are larger than all a_i 's. If non-empty, B may contain or not a least element. This describes three isomorphism types.) As an exercise the reader can generalize this example and exhibit for each $n > 2$ a complete theory with exactly n countable models. **EXERCISE**

10.4.3 No complete theory has exactly two countable models.

The following simple result is somewhat surprising. The proof is short because it builds on many concepts and results of the previous chapters.

PROPOSITION Let L be countable. If T is not ω -categorical then it has at least three non-isomorphic countable models.

PROOF If T has uncountably many parameter-free types then, since at most countably many of these can be realized in a countable model, T has uncountably many pairwise non-isomorphic models. So we can assume that T has only countably many parameter-free types (and consequently, countably many types over any finite set). So T has a countable atomic models over every finite set. Let M be an atomic model. Since T is not ω -categorical there is a tuple \bar{b} which is not isolated over \emptyset . Let N be a model atomic over \bar{b} . Clearly, a model containing \bar{b} cannot be isomorphic to M . There is a tuple \bar{c} which is not isolated over \bar{b} otherwise T would be ω -categorical over \bar{b} which, by the observation in 10.2.4 above, is equivalent to saying that T is ω -categorical. By (the argument of) Proposition 6.1.7 above there is a countable saturated model K containing $\bar{b}\bar{c}$. As K contains \bar{b} it cannot be isomorphic to M so we need only to show that K cannot be isomorphic to N . No model containing \bar{c} can be isomorphic to N over \bar{b} , so it suffices to show that the existence of an isomorphism between K and N implies the existence of a isomorphism over \bar{b} . Indeed if K and N are isomorphic then N would be saturated as well. Two countable saturated models are isomorphic over every common finite subset.

10.4.4 An exercise.

Let L be countable. Suppose T all countable models of T are homogeneous. If T is not ω -categorical then the number of non-isomorphic countable models is either \aleph_0 or 2^{\aleph_0} . (The dichotomy: either T is ω -categorical or it has infinitely many countable model, uses the ideas of the proof of 10.4.3 above. To prove that if there are uncountably many countable models then there is a continuum of such models, it is necessary to construct a binary tree of formulas as in 10.1.4 above.) **EXERCISE**

Chapter 11

Generic structures

From Chapter 6 we are used work inside a monster model. This setting simplifies the notation and offers a more tangible picture. E.g. extending a model M (otherwise a syntactical operation) can be thought as adding to M objects of \mathcal{U} . Also, we can safely identify each model M with its domain. Clearly, to be sure that in this way we are not excluding interesting objects, we need to require that \mathcal{U} is large enough to contain (an isomorphic copy of) all structures we may be interested in. So far, the class in question has been the class of models of some complete theory T . In this chapter we consider a more general case. This is an occasion to reconsider the notions of universality, saturation, and homogeneity

Notation and global assumptions.

The signature L is fixed, so **structure** below stands for structure of signature L .

11.1 Generic structures

Countable generic structures have first been introduced by logicians. A particular generic structure, the *countable random graph*, has been discovered independently by graph theorists, in fact, this has interesting combinatorial and probabilistic properties (it is also known as *Rado graph*). Another name for generic structures is *universal homogeneous structures*. Universality and homogeneity are two aspects of genericity that are also discussed in this chapter. The term *random* is also used for generic, in fact the construction of countable generic structures can be interpreted as the result of a stochastic processes. The term *generic* originates from a topological interpretation of the construction (the construction can be interpreted as forcing). However probability and/or topology do not add much insight, at least not at this stage. So our presentation is very near to Fraïsse's original construction: generic structures are simply the result of a straightforward diagonalization procedures.

11.1.1 Models and morphisms.

We fix an elementary class \mathcal{K} of structures (that is, two elementary equivalent structures are either both in \mathcal{K} or both not in \mathcal{K}). Below, by **model** we understand an element of \mathcal{K} . We fix also a class \mathcal{F} of (partial) maps between models. Below, by **morphism** we understand an element of \mathcal{F} . When a morphism is total we call it an **embedding**. We assume the following axioms hold for $\langle \mathcal{K}, \mathcal{F} \rangle$:

- F1** the empty map between any pair of models is a morphism;
- F2** every elementary map is a morphism and every morphisms is a partial isomorphisms;
- F3** the inverse of a morphism is a morphism the composition of two morphisms is a morphism;
- F4** the restriction of a morphism is a morphism, that is, if $F : M \rightarrow N$ is a morphism and $H \subseteq F$ then $H : M \rightarrow N$ is also a morphism;

We write $M \leq N$ when M is a substructure of N and the identity map $\text{id}_M : M \rightarrow N$ is an embedding of M into N in the sense of the definition above. We say that **M is a submodel of N** .

We say that the morphism $F' : M' \rightarrow N'$ **extends** $F : M \rightarrow N$ if F' extends F as a function, $M \leq M'$, and $N \leq N'$.

11.1.2 Submodels.

By axiom (F2), submodels are always substructures and elementary substructures are submodels. From (F3) it follows that \leq is a transitive relation; we also have that, if $N, K \leq M$ and $K \subseteq N$, then $K \leq N$. In fact, it suffices to compose $\text{id}_K : K \rightarrow G$ with the inverse of $\text{id}_N : N \rightarrow G$.

If $F : M \rightarrow N$ is an embedding then $F[M] \leq N$. In fact, as $F^{-1} : F[M] \rightarrow M$ is an isomorphism, by (F2) it is morphism so, composing it with $F : M \rightarrow N$, we obtain that the natural embedding of $F[M]$ into N is also a morphism.

11.1.3 Generic structures.

Let $\langle \mathcal{K}, \mathcal{F} \rangle$ be like above. Let G be an infinite model, cardinals $< |G|$ will be called **small**. We say that G is a **generic model** if it satisfies the following:

- G** every morphism $F : M \rightarrow G$, such that $|M| \leq |G|$ and F has small cardinality, extends to an embedding of M into G .

When \mathcal{K} is the class of all structure of some fixed characteristic and \mathcal{F} is the class of partial isomorphisms between models, then we may also say that G is a **random structure**.

11.1.4 An example.

Let T be any complete theory without finite models. Let \mathcal{K} be the class of all structures of a given complete theory and let \mathcal{F} be the class of elementary maps between models. Then the generic models are exactly the saturated model of T .

11.1.5 Universal structures.

Requirement (G) above has a weaker form that is interesting on its own. Let $\langle \mathcal{K}, \mathcal{F} \rangle$ be as above. We say that G is a **universal structure** if

U every model M such that $|M| \leq |G|$ embeds into G .

Generic models are universal: it suffices to extend the empty morphism to an embedding.

11.1.6 Homogeneous structures.

Generic models are large not only from the point of view of submodels but also from the point of view of automorphisms. Let $\langle \mathcal{K}, \mathcal{F} \rangle$ be as above class. We say that G is a **homogeneous structure** if

H every morphism $F : G \rightarrow G$ of small cardinality extends to an automorphism of G .

Generic models are homogeneous; this follows from Proposition 11.1.8 below. In 11.1.9 we also show that universality and homogeneity together imply genericity. So generic structures are also called **universal-homogeneous** structures.

When \mathcal{K} is the class of all structures of a given complete theory and \mathcal{F} contains only the elementary maps between models, this definition coincides with that given in Section 6.2 above. In general this is a stronger notion of homogeneity. In this chapter we say **elementarily homogeneous** for the notion introduced in Section 6.2 above.

11.1.7 Uniqueness

Now, via a back-and-forth construction, we show that generic structures (if they exist) are unique up to cardinality.

PROPOSITION Assume L is countable. Fix $\langle \mathcal{K}, \mathcal{F} \rangle$ as above. Then any two generic models of the same cardinality are isomorphic.

PROOF Let M and N be generic. First we give the proof in the case M and N are countable. Fix some $\{a_i\}_{i \in \omega}$ and $\{b_i\}_{i \in \omega}$ of M , respectively, N . The isomorphism $F : M \rightarrow N$ is the limit of a chain of morphisms $F_i : M \rightarrow N$ that we now define. Let F_0 be the empty map. Assume that F_i has been defined and that it is finite. Suppose first that i is even. By genericity, there is an embedding $H : M \rightarrow N$ that extends $F_i : M \rightarrow N$. Let F_{i+1} be the restriction of H to $\text{dom } F_i + a_i$. Suppose now that i is odd and proceed symmetrically: let $H : N \rightarrow M$ be an embedding that

extends $F_i^{-1} : N \rightarrow M$. Let F_{i+1} be the inverse of restriction of H to $\text{rng } F_i + b_i$. This completes the construction of $F : M \rightarrow N$. It is clear that in the limit we obtain an isomorphism: even stages work for totality, odd stages take care of surjectivity. This completes the proof in the countable case.

To generalize this construction we must find a way to get through the limit stages. Simply taking the limit is not enough: no axiom guarantees that isomorphisms are closed under taking the limit. So, suppose M and N have uncountable cardinality κ and fix some enumerations $\{a_\alpha\}_{\alpha < \kappa}$ and $\{b_\alpha\}_{\alpha < \kappa}$ of M , respectively, N . Again, let F_0 be the empty map. First we consider the successor stages: assume that F_α has been defined and that it has cardinality $< \kappa$. By genericity, there is an embedding $H : M \rightarrow N$ that extends $F_\alpha : M \rightarrow N$. Let $F_{\alpha+1}$ be the restriction of H to some elementary submodel of M containing $\text{dom } F_\alpha + a_\alpha$ and of cardinality $< \kappa$. Since we have assumed that $|F_\alpha| < \kappa$, this model exists by Löwenheim-Skolem (recall, L is countable). The function $F_{\alpha+1}$ has small cardinality, so the induction hypothesis is preserved. At odd stages we act symmetrically. Now suppose that α is limit and that F_β is defined for every $\beta < \alpha$. Let H be the union of F_β for $\beta < \alpha$. Observe that $H : M \rightarrow N$ is elementary, in fact, H maps isomorphically

$$\text{dom } H = \bigcup_{\beta < \alpha} \text{dom } F_\beta \quad \text{into} \quad \text{rng } H = \bigcup_{\beta < \alpha} \text{rng } F_\beta$$

and, by the elementary chain lemma, these are elementary substructures of M respectively N . So, by (F2) above, $H : M \rightarrow N$ is a morphism. We define F_α to be the restriction of H to

$$\{a_\beta : \beta < \alpha\} \cup \{H^{-1}(b_\beta) : \beta < \alpha\}$$

So $a_\beta \in \text{dom } F_\alpha$ and $b_\beta \in \text{rng } F_\alpha$ for every $\beta < \alpha$. (We cannot simply define F_α to be H because we did not assume κ to be a regular cardinal.)

11.1.8 Corollaries and generalizations

In the proof above we construct an isomorphism starting from the empty map. It is clear that the starting point can be an arbitrary morphism $F : M \rightarrow N$ of small cardinality. So we obtain that: if M and N are generic, then any isomorphism of small cardinality between M and N extends to an isomorphism. Observe also that if the construction above is applied to a pair of generic structures of different cardinality, say $|M| < |N|$, then the construction stops after $|M|$ stages and we only have an elementary embedding of M into N . To sum up we have the following.

PROPOSITION Every morphism between generic models M and N such that $|M| \leq |N|$ can be extended to an elementary embedding or, when M and N have the same cardinality, to an isomorphism between M and N . In particular, every morphism between generic models is an elementary map so, all generic models are elementary equivalent.

11.1.9 Generic = universal + homogeneous.

Using the back-and-forth method one can easily show that homogeneity and universality actually characterize genericity. Observe that this proposition is just a generalization of 6.2.18 above (you may think universality as a generalization of weak saturation).

PROPOSITION Let $\langle \mathcal{K}, \mathcal{F} \rangle$ be an as above. The following are equivalent:

- 1 G is a generic structure; and
- 2 G is universal and homogeneous.

PROOF The direction 1 \Rightarrow 2 is clear by what observed in 11.1.8 above. To prove the direction 2 \Rightarrow 1 assume G is universal and homogeneous, let M be a model and $F : M \rightarrow G$ be a morphism. We need to show that it has an extension to an embedding. By universality, there is an embedding $H : M \rightarrow G$, though this need not be an extension of F . By hypothesis there is an automorphism $L : G \rightarrow G$ that extends $FH^{-1} : G \rightarrow G$, so the embedding $LH : M \rightarrow G$ is the required extension of $F : M \rightarrow G$.

11.2 Saturation and quantifier elimination

When generic models are saturated, their theory is simplified. In fact, inside a saturated generic model the relation \leq coincides with \preceq .

11.2.1 If one generic is saturated, they all are

The first proposition we prove shows that saturation of a generic model is never accidental but is an intrinsic property of the theory of generic models.

PROPOSITION Fix $\langle \mathcal{K}, \mathcal{F} \rangle$ such that there exist generic model of arbitrarily large cardinality. Let G be any generic model. The following are equivalent.

- 1 some generic model is saturated;
- 2 every generic model is saturated; and
- 3 every saturated model of $\text{Th}G$ is generic.

PROOF We prove 1 \Rightarrow 2 first. Let G be a saturated generic model and let M be a generic model, then M is elementarily homogeneous, so by 6.2.18 it suffices to prove that M is weakly saturated. Suppose first that $|G| \leq |M|$; by 11.1.8 above, we can assume that $G \preceq M$. Since G is weakly saturated then M is also weakly saturated. Now suppose $|M| < |G|$; let $p(\bar{x})$ be a parameter-free type, and let \bar{b} be a realization of $p(\bar{x})$ in G . Let $N \preceq G$ be a countable model containing \bar{b} . Then there is an embedding $F : N \rightarrow M$. Then $F : G \rightarrow M$ is morphism so, by 11.1.8 it is elementary. Then $F\bar{b}$ realizes $p(\bar{x})$ in M .

To prove 2 \Rightarrow 3 assume M is a saturated model of $\text{Th}G$. Without loss of generality

we can assume $|M| \leq |G|$, otherwise we simply replace G with some larger generic model. Let $F : N \rightarrow M$ be a morphism, where $|N| \leq |M|$ and $|F| < |M|$. Since G is saturated we can assume that $M \preceq G$ and, by genericity, we assume that $N \leq G$. Then $F : G \rightarrow G$ is morphisms and, by 11.1.8 above, it is an elementary map. Fix N' with the same cardinality as N and such that $N \subseteq N' \preceq G$. Then $F : N' \rightarrow M$ is an elementary map and, by the saturation of M , it extends to an elementary embedding $H : N' \rightarrow M$. By what observed in 11.1.2 above, $N \leq N'$. So the required embedding of N into M is obtained composing $\text{id}_N : N \rightarrow N'$ with $H : N' \rightarrow M$.

Finally, the implication 3 \Rightarrow 1 is clear.

11.2.2 Generics are saturated $\leftrightarrow \leq$ coincides with \preceq .

Finally we show that the saturation of generic models is equivalent to a property very close to quantifier elimination for $\text{Th}G$ (when \mathcal{F} is the class of partial isomorphisms, \leq is the relation of substructure). This property is *quantifier separation*, as we see in the next section.

PROPOSITION Fix $\langle \mathcal{K}, \mathcal{F} \rangle$ such that there exist generic model of arbitrarily large cardinality. Let G be any generic model. The following are equivalent.

- 1 G is saturated; and
- 2 for every model $M \models \text{Th}G$, if $M \leq G$ then $M \preceq G$.

PROOF To prove 1 \Rightarrow 2 assume G is saturated and let $M \leq G$ be arbitrary. As $M \models \text{Th}G$, by saturation there is an elementary embedding $F : M \rightarrow G$. So $F : G \rightarrow G$ is a morphism hence, by 11.1.8 above, it is an elementary map. So, from $F[M] \preceq G$ it follows that $M \preceq G$. To prove 2 \Rightarrow 1 fix an arbitrary type $p(x)$ over some set $A \subseteq G$ of small cardinality. Let M be a model realizing $p(x)$. By genericity we can embed M into G so, up to isomorphism we can assume that $M \leq G$. Then, by hypothesis $M \preceq G$, hence G realizes $p(x)$.

11.3 Amalgamation classes

Now we introduce a condition that ensures the existence of generic structures.

11.3.1 Amalgamation classes.

Let \mathcal{K} be a class of models and let \mathcal{F} be a class of morphisms. We say that $\langle \mathcal{K}, \mathcal{F} \rangle$ is an **amalgamation class** if it meets the following conditions:

- A every morphism extends to an embedding. That is, every morphism $F : M \rightarrow N$ has an extension $F' : M \rightarrow N'$ that is an embedding of M into N' . (Cf. the definition of *embedding* and of *extension* in 11.1.1 above.)

This axiom is called **amalgamation property**.

In practice, to prove amalgamation of some give class it is always convenient to assume that M and N in (A) intersect on a common substructure K and that $F : M \rightarrow N$ is the identity map on K . Then we work to extend N to a superstructure of both M and N . Then $F' : M \rightarrow N'$ is the identity map on M . We describe this by saying that **N' amalgamates M and N** .

11.3.2 Inductive classes.

Often we need to take the union of infinite chains of models or of morphisms, then it is essential to assume the following closure property:

II the union of a chain of morphisms is a morphism extending every element of the chain.

When this property holds we say that **$\langle \mathcal{K}, \mathcal{F} \rangle$ is an inductive class**. (N.B. in this section, *chains* consists of *extensions* as defined in 11.1.1 above.)

11.3.3 An exercise.

Let $\langle \mathcal{K}, \mathcal{F} \rangle$ be an inductive amalgamation class. Prove (using a back-and-forth construction) that every morphism extends to an isomorphism. That is, every morphism $F : M \rightarrow N$ has an extension $F' : M' \rightarrow N'$ that is an isomorphism between M' and N' . EXERCISE

11.3.4 The construction of generic models.

Given an amalgamation class $\langle \mathcal{K}, \mathcal{F} \rangle$, we construct a generic structure G by amalgamating all models over all possible morphisms. This we do in turn, so the only difficulty is to fix an ordering that gives to every model and every map a chance.

PROPOSITION Fix an inductive amalgamation class $\langle \mathcal{K}, \mathcal{F} \rangle$. Assume that, up to isomorphism, there are at most λ morphisms with domain and codomain of small (i.e. $< \lambda$) cardinality. Then there is a generic model.

PROOF We define a chain $\langle G_\alpha : \alpha < \lambda \rangle$ of models of small cardinality. The generic structure G will be the limit of this chain. Fix a set of morphisms \mathcal{H} that contains an isomorphic copy of every morphism with domain and codomain of small cardinality. Fix a well-ordering of \mathcal{H} . The starting point of the construction, G_0 , can be any model of small cardinality. At stage α let $H : M \rightarrow N$ be (a copy of) the least morphism of \mathcal{H} such that $N \leq G_\alpha$ but does not extend to an embedding of M into G_α . Let $F : M \rightarrow G_{\alpha+1}$ be an extension of $H : M \rightarrow G_\alpha$ to an embedding. This completes the definition of G .

Let M be a model of cardinality $\leq \lambda$. Let $H : M \rightarrow G$ be a morphism such that

$|H| < \lambda$. The domain of definition of H is contained in some $N \preceq M$ of cardinality $< \lambda$ and, since λ is regular cardinal, the range of H is contained in G_α for some α . Then $H : N \rightarrow G_\alpha$ is a morphism of those contained in \mathcal{H} . So there is an extension that embeds N into G . As N can be fixed to contain arbitrary elements of M , in the limit M embeds into G .

11.3.5 Quantifier-free amalgamation lemma.

Let \mathcal{K} contain all structures of a given signature and characteristic (i.e. isomorphism type of the substructure generated by the empty set) and let \mathcal{F} contain all partial isomorphisms between models. We claim that $\langle \mathcal{K}, \mathcal{F} \rangle$ is an inductive amalgamation class. Only the amalgamation property requires a proof but this is easy: there is a straightforward way to amalgamate structures: this is called **free amalgamation of M and N over F** .

PROPOSITION Let M and N be two structures. Every partial isomorphism $F : M \rightarrow N$ extends to an embedding $F' : M \rightarrow N'$.

PROOF The amalgamation is constructed by hand in the most obvious and direct way; we only sketch the proof and leave the details to the reader. Without loss of generality we can assume that F is an isomorphism between a substructure of M and a substructure of N (see Exercise 3.2.3 above). Up to renaming, we can assume that M and N actually coincide on a common substructure and that F is just the identity on this substructure. Now, the domain of N' is simply the union of M and N . The relations of N' are also the union of the relations on M and N (so no relation holds on tuples that mix elements of N with elements of M). Finally we extend the functions of M and N to N' in an arbitrary way.

11.3.6 Elementary amalgamation lemma.

Let \mathcal{K} contain all models of a given complete theory and let \mathcal{F} contain all elementary maps between models. We claim that $\langle \mathcal{K}, \mathcal{F} \rangle$ is an inductive amalgamation class. Only the amalgamation property is non obvious (inductivity is the elementary chain lemma). The proof in this case it is much less constructive then for the quantifier-free case.

PROPOSITION Every elementary map $F : M \rightarrow N$ has an elementary extension to an isomorphisms between structures of cardinality $\leq |LMN|$.

PROOF Let M and N be infinite structures (every elementary map between finite structures is an isomorphism, so there is nothing to prove in this case). We show that there is an elementary extension of $F : M \rightarrow N$ to an elementary embedding $F' : M \rightarrow N'$ where $N \preceq N'$. Let $p(\bar{x})$ be the diagram of M over $\text{dom } F$, that is, we fix an infinite tuple \bar{a} with as range M and let $p(\bar{x})$ be the set of formulas $\varphi(\bar{x})$ over the domain of definition of F that are realized by \bar{a} in M . Since $p(\bar{x})$

is, by construction, consistent in M , and elementary maps map consistent formulas to consistent formulas, then $Fp(\bar{x})$ is a set of formulas over the range of F that is finitely consistent in N . Let N' be an elementary extension of N that realizes $Fp(\bar{x})$ and fix a realization \bar{c} . Let $F' : M \rightarrow N'$ be such that $F'\bar{a} = \bar{c}$.

11.3.7 Basic sets.

The two examples above can be generalized as follows. We fix a set δ of **basic formulas** with the following properties:

D1 at least all atomic formulas are basic; and

D2 basic formulas are closed under boolean connectives.

Fix a theory T complete for basic sentences. A class of models and morphisms is obtained taking as models the models of T and, as morphism, the maps $F : M \rightarrow N$ between models that preserve truth of basic formulas, that is, for every basic formula $\varphi(\bar{x})$ and for every $\bar{a} \subseteq \text{dom } F$ if $M \models \varphi(\bar{a})$ then $N \models \varphi(F\bar{a})$.

The axiom above are clearly satisfied: (F3) holds because (D1) guarantees the existence of the inverse map (because equality is basic) and (D2) closure under negation. So, when Δ is the set of parameter- and quantifier-free sentences then \mathcal{F} is the set of partial morphisms. When Δ is the set of all parameter-free formulas then \mathcal{F} is the set of elementary maps. The relation $M \leq N$ becomes the substructures relation, respectively, the elementary substructure relation.

11.3.8 Separation of quantifiers.

Fix a set of basic formulas Δ as above. Let T be a theory. We say that T admits **separation of quantifiers** if for every Δ -formulas $\varphi(\bar{x}\bar{y})$ and $\psi(\bar{x}\bar{z})$ such that

$$T \vdash \varphi(\bar{x}\bar{y}) \rightarrow \psi(\bar{x}\bar{z})$$

there is a basic formula $\vartheta(\bar{x})$ such that

$$T \vdash \varphi(\bar{x}\bar{y}) \rightarrow \vartheta(\bar{x}) \quad \text{and} \quad T \vdash \vartheta(\bar{x}) \rightarrow \psi(\bar{x}\bar{z}).$$

Observe that in the implications above we can prefix $\varphi(\bar{x}\bar{y})$ with $\exists\bar{y}$ and $\psi(\bar{x}\bar{z})$ with $\forall\bar{z}$. So, another good name for this property could be **quantifier interpolation**. We call the formula $\vartheta(\bar{x})$ a **Δ -interpolant between $\exists\bar{y}\varphi(\bar{x}\bar{y})$ and $\forall\bar{z}\psi(\bar{x}\bar{z})$** .

11.3.9 Quantifier separation \leftrightarrow amalgamation.

The following result generalizes both amalgamation lemmas above. Fix a set Δ of basic formulas. Let T be a theory without finite models that decides all basic sentences. Let \mathcal{K} be the class of models of T and let \mathcal{F} be the class of Δ -elementary maps between models. Quantifier separation is equivalent to the amalgamation property for $\langle \mathcal{K}, \mathcal{F} \rangle$.

PROPOSITION Let \mathcal{K} and \mathcal{F} be as above. Then the following are equivalent

1 T has separation of quantifiers, and

2 $\langle \mathcal{K}, \mathcal{F} \rangle$ is an amalgamation class.

PROOF We prove **1** \Rightarrow **2** first. Without loss of generality we can assume F is the identity map on some $K \leq M, N$. Let \bar{a}, \bar{b} and \bar{c} be infinite tuples that enumerate $K, M \setminus K$ and $N \setminus K$. Let $p(\bar{x}\bar{y})$ be the set of parameter-free basic formulas $\varphi(\bar{x}\bar{y})$ such that $\varphi(\bar{a}\bar{b})$ holds in M . Let $q(\bar{x}\bar{z})$ be the set of parameter-free basic formulas $\psi(\bar{x}\bar{z})$ such that $\psi(\bar{a}\bar{c})$ holds in N . Clearly it suffices to find a model of T that realizes $p(\bar{x}\bar{y}) + q(\bar{x}\bar{z})$. Suppose not then, by compactness,

$$T \vdash \varphi(\bar{x}\bar{y}) \rightarrow \neg\psi(\bar{x}\bar{z}).$$

For some formulas $\varphi(\bar{x}\bar{y})$ and $\psi(\bar{x}\bar{z})$ in $p(\bar{x}\bar{y})$, respectively, $q(\bar{x}\bar{z})$. Then for some basic formula $\vartheta(\bar{x})$ we have

$$T \vdash \varphi(\bar{x}\bar{y}) \rightarrow \vartheta(\bar{x}) \quad \text{and} \quad T \vdash \vartheta(\bar{x}) \rightarrow \neg\psi(\bar{x}\bar{z}).$$

Since $\exists\bar{y}\varphi(\bar{a}\bar{y})$ holds in M , then $\vartheta(\bar{a})$ holds in M . Since $K \leq M, N$ then $\vartheta(\bar{a})$ holds in N . Hence $\psi(\bar{a}\bar{z})$ is inconsistent. A contradiction.

For the converse, we prove **2** \Rightarrow **1** so assume T does not admit quantifier separation. Fix two basic formulas such that $T \vdash \varphi(\bar{x}\bar{y}) \rightarrow \psi(\bar{x}\bar{z})$ but there is no Δ -interpolant. We claim that there is a parameter-free complete Δ -type $p(\bar{x})$ such that both $\varphi(\bar{x}\bar{y})$ and $\neg\psi(\bar{x}\bar{z})$ are consistent with $T + p(\bar{x})$. The claim is proved below; we first show how proposition follows from the claim. Let M and N be models of T realizing $\varphi(\bar{x}\bar{y}) + p(\bar{x})$, respectively, $\neg\psi(\bar{x}\bar{z}) + p(\bar{x})$. Let \bar{a} be a realization of $\exists\bar{y}\varphi(\bar{x}\bar{y}) + p(\bar{x})$ in M and let \bar{c} be a realization of $\exists\bar{z}\neg\psi(\bar{x}\bar{z}) + p(\bar{x})$ in N . Let F map \bar{a} to \bar{c} , then $F : M \rightarrow N$ is a morphism (i.e. a Δ -elementary map) but has no extension to an embedding $F : M \rightarrow N'$, otherwise N' would satisfy $\varphi(\bar{c}\bar{y})$ but $\exists\bar{y}\varphi(\bar{c}\bar{y})$ implies $\forall\bar{z}\psi(\bar{c}\bar{z})$. Then also N would model $\forall\bar{z}\psi(\bar{x}\bar{z})$, a contradiction.

Now we prove the claim. Fix an enumeration of all parameter-free basic formulas. The type $p(\bar{x})$ is the union of a chain of types. We begin with the empty type and take the union at limit stages. At stage $\alpha + 1$ we add to $p_\alpha(\bar{x})$ either $\xi_\alpha(\bar{x})$ or $\neg\xi_\alpha(\bar{x})$, so in the end we obtain a complete type. As induction hypothesis we require that there is no basic formula $\vartheta(\bar{x})$ such that

$$T + p_\alpha(\bar{x}) \vdash \varphi(\bar{x}\bar{y}) \rightarrow \vartheta(\bar{x}) \quad \text{and} \quad T + p_\alpha(\bar{x}) \vdash \vartheta(\bar{x}) \rightarrow \psi(\bar{x}\bar{z})$$

This assumption holds by hypothesis at stage 0 and is obviously preserved at limit stages. It is immediate that either one of the definitions $p_{\alpha+1}(\bar{x}) = p_\alpha(\bar{x}) + \xi_\alpha(\bar{x})$ or $p_{\alpha+1}(\bar{x}) = p_\alpha(\bar{x}) + \neg\xi_\alpha(\bar{x})$ preserve the induction hypothesis. This proves the claim, hence the proposition.

11.3.10 An exercise.

Let L be relational. Let \mathcal{K} be the class of all structures of signature L . Let \mathcal{F} the class of partial isomorphism. Prove that every generic structure is isomorphic to a proper substructure of itself. EXERCISE

11.4 Examples

Dense linear orders and generic graphs are the examples of generic structures that we shall meet most frequently.

11.4.1 Countable dense linear orders.

Let L contains only the binary relation $<$ and let \mathcal{K} be the class of linear orders. The morphisms are the order-preserving maps between models. The reader can easily check that $\langle \mathcal{K}, \mathcal{F} \rangle$ is an inductive amalgamation class. So there are generic structures. We claim that any generic structure G is a dense linear orders without endpoints. We check density: let $a_1 < a_3$ be elements of G we claim there is a_2 in G such that $a_1 < a_2 < a_3$. Let $c_1 < c_2 < c_3$ be the elements of a model of cardinality 3. The map $c_1 c_3 \mapsto a_1 a_3$ is a partial isomorphism. So, by the definition of generic structure, it extends to an embedding. Now let a_2 be the image of c_2 under this embedding. In a similar way, the reader can check that G has no endpoints.

11.4.2 A non-example

Let $\langle \mathcal{K}, \mathcal{F} \rangle$ be like above. Let Z be the set of integers with the usual order. Show that Z is universal but it is not homogeneous. EXERCISE

11.4.3 Dense linear orders with a random color.

Let L contain a binary relation $<$ and a unary relation r (that is, a set). Let \mathcal{K} contain all finite structures where $<$ is interpreted in a linear order. Show that there are some homogeneous structures that are not generic. Axiomatize the theory of the generic structure. EXERCISE

11.4.4 The countable random graph.

Let L be the language of graphs. Let \mathcal{K} contain all graphs. The reader can easily check that \mathcal{K} is an inductive amalgamation class (free amalgamation suffices). The reader can check as an exercise that a countable graph G is a random graph if and only if for every two finite subsets A and C of G there is a vertex $b \in G$ which is linked to all elements of A and no element of B . EXERCISE

11.4.5 The triangle free random graph.

Let L be the language of graphs and let M be a structure. A triangle-free graph is a structure that does not contains any elements a, b , and c such that $r(ab) \wedge r(bc) \wedge r(ca)$. Let \mathcal{K} contain all triangle-free graphs. The reader can check as an exercise that \mathcal{K} is an inductive amalgamation class. EXERCISE

Chapter 12

Definability and almost-definability

This is the first of two chapters on imaginary elements. A good understanding of first-order definability requires definable sets to be regarded as sort of elements: in this context definable sets are called *imaginary elements*.

There is a translation, systematic but non-trivial, of the result in Section 7.1 to results about definable sets. In Section 7.1 we introduced the notion of algebraic element and now we want to generalize it to the *second-order* objects of the universe: the definable sets.

Notation and global assumptions.

As in Chapter 6 above.

12.1 Definability and fixed points

There are two contexts where we use word *definable*: we say that the tuple \bar{b} is definable over A if $\varphi(\bar{b}) \wedge \exists^{=1} \bar{x} \varphi(\bar{x})$ holds for some A -formula $\varphi(\bar{x})$; and we say that the set $\mathcal{D} \subseteq \bar{U}$ is definable over A if it is of the form $\varphi(\bar{U})$ for some A -formula $\varphi(\bar{x})$. Our first goal is to show that these two concepts have in common something more than the name.

12.1.1 Automorphisms and orbits.

Let F be an automorphism of the universe: there is an obvious action of F on subsets of the universe. Let $\mathcal{D} \subseteq \bar{U}$; the set

$$O_A \mathcal{D} := \left\{ F\mathcal{D} : F \text{ is an } A\text{-automorphism} \right\}$$

is called the **orbit of \mathcal{D} over A** . When \mathcal{D} is a definable set we have a second useful way of representing the orbit of \mathcal{D} . Recall that in Paragraph 3.2.6 above we have defined an action of F on formulas: $\varphi(\bar{x})$ maps to $F\varphi(\bar{x})$, that is, the formula obtained replacing the parameters of $\varphi(\bar{x})$ with their image under F . It is easy to see that if $\varphi(\bar{x})$ defines \mathcal{D} then $F\varphi(\bar{x})$ defines $F\mathcal{D}$. It follows that the sets in the orbit of a definable set are definable. We may read the expression $F\varphi(\bar{U})$ in two ways: the set defined by $F\varphi(\bar{x})$ and the image of $\varphi(\bar{U})$ under F . By the observation above the ambiguity is harmless.

12.1.2 Some remarks.

In this chapter \mathcal{D} will generally denote a definable set. We may use the expression $\bar{x} \in \mathcal{D}$ in formulas: we read this as $\delta(\bar{x})$, where $\delta(\bar{x})$ is any definition of \mathcal{D} .

12.1.3 Definability and automorphisms.

If we replace \mathcal{D} with $\{b\}$ the equivalence $1 \Leftrightarrow 3$ below translates as follows: b is definable over A if and only if every A -isomorphism fixes b ; something we already know from 7.1.15 above.

PROPOSITION Let \mathcal{D} be a definable set. The following are equivalent:

- 1 \mathcal{D} is definable over A ;
- 2 if $\bar{a} \equiv_A \bar{c}$ then $\bar{a} \in \mathcal{D} \leftrightarrow \bar{c} \in \mathcal{D}$; and
- 3 every A -automorphism fixes \mathcal{D} setwise.

PROOF The equivalence $2 \Leftrightarrow 3$ is clear: if F is a A -automorphism which does not fix \mathcal{D} setwise, there is an $\bar{a} \in \mathcal{D}$ and $\bar{a} \notin F\mathcal{D}$. So either $\bar{a} \in \mathcal{D}$ and $F^{-1}\bar{a} \notin \mathcal{D}$. Then $\bar{a} \in \mathcal{D} \leftrightarrow F^{-1}\bar{a} \in \mathcal{D}$. But clearly $\bar{a} \equiv_A F^{-1}\bar{a}$. The converse is a consequence of homogeneity.

The implication $1 \Rightarrow 2$ is immediate, so only $2 \Rightarrow 1$ requires a proof. Assume 2 and observe that for every complete A -type either $p(\bar{U})$ is contained in \mathcal{D} or it is disjoint of \mathcal{D} . Since \mathcal{D} is definable we obtain

$$\mathcal{D} = \bigcup_{\psi(\bar{U}) \subseteq \mathcal{D}} \psi(\bar{U}) \quad \text{and} \quad \mathcal{U} \setminus \mathcal{D} = \bigcup_{\psi(\bar{U}) \subseteq \mathcal{U} \setminus \mathcal{D}} \psi(\bar{U}).$$

it follows that both \mathcal{D} and $\mathcal{U} \setminus \mathcal{D}$ are type-definable over A . So, by Exercise 4.4.2 above, we conclude that \mathcal{D} is definable over A .

12.1.4 Addenda.

To the three equivalents in the definability above theorem we can add the following two: 4 $\mathcal{D} \subseteq F\mathcal{D}$ for every A -automorphism F of the universe; and 5 $F\mathcal{D} \subseteq \mathcal{D}$ for every A -automorphism F of the universe. In fact, since the inverse of a A -

automorphisms is again a A -automorphism, we immediately obtain the equivalence with item 3 of the theorem above.

12.1.5 An exercise.

Suppose \mathcal{D} is merely type-definable, that is, \mathcal{D} is the intersection of a (small) family of definable sets. Clearly (2) and (3) of Proposition 12.1.3 above are still equivalent. Modify (3) so that the proposition holds under this weaker hypothesis. EXERCISE

12.1.6 An exercise.

If $\psi(\bar{z}\bar{x})$ is a parameter-free formula, then every model M defining $\psi(\bar{c}\bar{U})$ contains a tuple \bar{a} such that $\psi(\bar{a}\bar{x})$ defines $\psi(\bar{c}\bar{U})$. (It is essential that M is a model.) EXERCISE

12.1.7 An exercise.

Let \mathcal{D} be a subset of the universe such that property (\sharp) of 12.1.2 holds. The reader can prove the following corollary of 12.1.3 as an exercise. If \mathcal{D} has a small orbit then \mathcal{D} is definable (this holds if *small* is replaced with $< 2^\kappa$ but the proof is harder). Moreover \mathcal{D} has a definition with parameters in \mathcal{D} .

12.1.8 An interpolation theorem.

The next fact is not needed in the following, so we propose it as an exercise. It is easy to see that the definability theorem above is a consequence of it.

PROPOSITION Let \mathcal{D} and \mathcal{C} be definable sets (of the same arity). Suppose that $F[\mathcal{D}] \subseteq \mathcal{C}$. Then there is an A -formula $\varphi(\bar{x})$ such that $\mathcal{D} \subseteq \varphi(\bar{U}) \subseteq \mathcal{C}$. The formula φ is called an **interpolant**. EXERCISE

12.2 Finite orbits of sets

In Paragraph 7.1.13 we obtained two intrinsic characterizations of algebraic elements: **1** algebraic elements are those that have finite orbits under the automorphisms that fix the parameters; and **2** algebraic elements are those that are contained in every model containing the parameters. In the next section we will see how to generalize algebraicity at the second-order. But we already know how to lift the two other notions (almost-in and finite orbits) at the level of definable sets. In this section we prove that we obtain two equivalent notions.

In this section we move from fixed points to finite orbits (of definable sets) and from definability to almost-definability. In the jargon of model theorists, the definability

theorem above considers the *stationary* case while now we are going to consider the *non-stationary* case. Informally this is known as the hard part of the theory.

12.2.1 Almost-definable sets.

We say that a set \mathcal{D} is **almost-definable over A** or simply **almost-over A** if it is definable over every model M that containing A . In particular, taking for \mathcal{D} the singleton of b we obtain that $\{b\}$ is almost-over A if and only if b is algebraic over A . So almost-over is the second-order version almost-in.

12.2.2 An example.

A straightforward example of non-definable but almost-definable sets comes from the theory of an equivalence relation $E(x, y)$ with n infinite equivalence classes. The automorphisms of the universe are exactly the permutations that respect $E(x, y)$. If $E(x, y)$ has more than one equivalence class, then there is an automorphism which permutes non-trivially these equivalence classes (by saturation all equivalence classes have cardinality κ). So none of these equivalence classes is definable over \emptyset . But every model intersects all equivalence classes. So these are definable over any model: they are almost-over \emptyset . In the next sections we shall see that this example is paradigmatic.

12.2.3 An almost-definability theorem.

Here we generalize to definable sets part of proposition 7.1.13 above. If we replace \mathcal{D} with $\{b\}$ the equivalence 1 \Leftrightarrow 3 below translates as follows: b is almost-in A if and only if b has a finite orbit under A -isomorphisms.

PROPOSITION Let \mathcal{D} be a definable set. The following are equivalent:

- 1** \mathcal{D} is almost-over A ;
- 2** the orbit of \mathcal{D} under A -automorphisms has small cardinality; and
- 3** the orbit of \mathcal{D} under A -automorphisms is finite.

PROOF To prove **1 \Rightarrow 2** assume that \mathcal{D} is almost-over A and fix an arbitrary model M containing A and of cardinality $|TA|$. Let F be an arbitrary A -automorphism of the universe. Observe that $F^{-1}M$ is also model containing A (cf. Exercise 3.2.5 above) so, by hypothesis, \mathcal{D} is definable over $F^{-1}M$. This is equivalent to saying that $F\mathcal{D}$ is definable over M . There are at most $|M|$ non-equivalent M -formulas. So the orbit of the set \mathcal{D} has cardinality at most $|M|$, hence it is small.

To prove the implication **2 \Rightarrow 3** suppose that $\delta(\bar{c}\bar{x})$ is a definition of \mathcal{D} , where $\delta(\bar{z}\bar{x})$ is a parameter-free and \bar{c} is arbitrary. Let $p(\bar{x})$ be the type of \bar{c} over A . Fix a sequence of κ tuples of variables \bar{x}_α with disjoint range and consider the sets of formulas, containing for every $\alpha < \beta < \kappa$,

$$p(\bar{x}_\alpha) + \exists \bar{w} \left[\delta(\bar{x}_\alpha \bar{w}) \leftrightarrow \delta(\bar{x}_\beta \bar{w}) \right].$$

If $|O_A \mathcal{D}| < \kappa$, this set cannot be consistent. Then a finite subset of it is inconsistent. It follows that the orbit of \mathcal{D} over A is finite.

Finally, to prove [3 \$\Rightarrow\$ 1](#) we show that some tuple of parameters \bar{d} such that $\delta(\bar{d} \bar{x})$ is equivalent to $\delta(\bar{c} \bar{x})$ can be found in any model M containing A . Define for each m a parameter-free formula

$$\vartheta_m(\bar{z}_1 \cdots \bar{z}_m) := \bigwedge_{1 \leq h < k \leq m} \exists \bar{x} \left[\delta(\bar{z}_h \bar{x}) \leftrightarrow \delta(\bar{z}_k \bar{x}) \right].$$

Let $p(\bar{x})$ be the A -type of \bar{c} . Since the orbit of $\delta(\bar{c} \bar{U})$ is finite, there is an n that is maximal such that

$$(*) \quad p(\bar{z}_1) + \cdots + p(\bar{z}_n) + \vartheta_n(\bar{z}_1 \cdots \bar{z}_n)$$

is consistent. As n is chosen maximal, by saturation there is a formula $\varphi(\bar{x})$ in $p(\bar{x})$ such that

$$(**) \quad \varphi(\bar{z}_1) + \cdots + \varphi(\bar{z}_{n+1}) + \vartheta_{n+1}(\bar{z}_1 \cdots \bar{z}_{n+1})$$

is inconsistent. Since $(*)$ above is consistent, we can fix some tuples $\bar{d}_1 \cdots \bar{d}_n$ in M that satisfy

$$\varphi(\bar{z}_1) + \cdots + \varphi(\bar{z}_n) + \vartheta_n(\bar{z}_1 \cdots \bar{z}_n)$$

We claim that $\delta(\bar{c} \bar{U})$ equals $\delta(\bar{d}_i \bar{U})$ for some i . In fact, if not then $\bar{d}_1 \cdots \bar{d}_n \bar{c}$ would satisfy $\vartheta_{n+1}(\bar{z}_1 \cdots \bar{z}_{n+1})$ and, since \bar{c} obviously satisfies $\varphi(\bar{z}_{n+1})$, we would have a solution of $(**)$ which was assumed to be inconsistent.

12.2.4 Finite orbits over models are trivial.

Assume the hypotheses and the notation of the almost-definability theorem above but substitute a model M for A . Then from the proposition above it follows that \mathcal{D} has a finite orbit under M -automorphisms if and only if \mathcal{D} is fixed setwise by M -automorphisms.

12.2.5 An exercise.

Let T be strongly minimal. Let $\varphi(x y)$ be an A -formula and let b be arbitrary. Suppose that $\varphi(\mathcal{U} b)$ is almost-definable over A . Prove that $\varphi(\mathcal{U} b)$ is definable over $\text{acl}(A + e)$ for every $e \notin \text{acl} A$.

EXERCISE

12.3 Finite equivalence relations

An easy but non-trivial example of almost-definable set has been obtained in Paragraph [12.2.1](#) using a definable equivalence relation with finitely many equivalence

classes. Here we show that this is not a coincidence, indeed almost-definable sets can be characterized syntactically using so-called finite equivalence relations. Formulas naturally partition the universe into two natural classes: true and false. Almost-definability can be better understood if we consider (definable) partitions which divide the universe into finitely many classes. This leads us to a fundamental concept: that of finite equivalence relation. We use it to give a syntactic characterization of almost-definability.

12.3.1 Finite equivalence relations.

We say that an A -formula $\varepsilon(\bar{x}, \bar{y})$ is a **finite equivalence relation (definable) over A** if it defines an equivalence relation which partitions \bar{U} into finitely many **cosets** (i.e. equivalence classes). The relevance of this notion is related to a fundamental property that the cosets of a finite equivalence relations $\varepsilon(\bar{x}, \bar{y})$ share with definable sets: every model defining $\varepsilon(\bar{x}, \bar{y})$ intersects all these cosets. In fact, the formula that claims the existence of n pairwise non-equivalent tuples –if true– is satisfied in every model. When n is maximal, any realization yields a representative for each coset.

12.3.2 Equivalence formulas.

Let $\delta(\bar{x})$ be a formula. If for some finite equivalence relation $\varepsilon(\bar{x}, \bar{y})$ over A we have

$$(\diamond) \quad \forall \bar{x} \bar{y} \left[\varepsilon(\bar{x}, \bar{y}) \rightarrow \left[\delta(\bar{x}) \leftrightarrow \delta(\bar{y}) \right] \right]$$

then we say that **$\delta(\bar{x})$ is an equivalence formula over A** or that it is an **equivalence A -formula** (the terminology is non-standard but it is temporary until we prove that equivalence A -formulas coincides with formulas that are almost-over A). In other words, $\delta(\bar{x})$ is an equivalence A -formula if it is equivalent to a disjunction of the form

$$(\diamond) \quad \bigvee_{i=1}^n \varepsilon(\bar{x}, \bar{a}_i),$$

where $\varepsilon(\bar{x}, \bar{y})$ is a finite equivalence relation over A and the tuples $\bar{a}_1 \cdots \bar{a}_n$ are arbitrary.

12.3.3 Equivalence formulas (continued).

To every formula $\delta(\bar{x})$ we naturally associate the equivalence relation $\delta(\bar{x}) \leftrightarrow \delta(\bar{y})$. This equivalence relation is finite: it has just two classes. When $\delta(\bar{x})$ is an A -formula we may substitute $\delta(\bar{x}) \leftrightarrow \delta(\bar{y})$ for $\varepsilon(\bar{x}, \bar{y})$ in (\diamond) above and obtain a trivially true formula. So equivalence A -formulas are a generalization of ordinary A -formulas. It is also useful to observe that, since every model containing A contains a representative of each coset of $\varepsilon(\bar{x}, \bar{y})$, then we can require that $\bar{a}_1 \cdots \bar{a}_n$ in (\diamond)

above belong to any particular model M containing A . It follows that equivalence formulas over A are almost-over A . This may be useful in various circumstances, for instance, to bound the cardinality of equivalence A -formulas: by the Löwenheim-Skolem theorem, this is at most $|T A|$. The main result of this section is the converse of this easy observation: equivalence A -formulas are exactly the formulas almost-over A .

12.3.4 Closures properties of equivalence formulas.

Very often we need to add redundant variables to formulas. This offers flexibility: when we consider a finite set of formulas we can assume that these depend on the same variables. The reader can check as an exercise that the same holds also finite equivalence relations, namely, if $\varepsilon(\bar{x}, \bar{y})$ is a finite equivalence relation then the relation $\varepsilon(\bar{x} \bar{w}, \bar{y} \bar{z})$ (we have padded redundant variables in $\varepsilon(\bar{x}, \bar{y})$) is also a finite equivalence relation. It follows that the conjunction of finitely many finite equivalence relations (possibly with different arity or depending on different variables) is again a finite equivalence relation. Consequently equivalence formulas are closed under conjunction, that is, if $\delta(\bar{x})$ and $\psi(\bar{y})$ are both equivalence A -formulas then $\delta(\bar{x}) \wedge \psi(\bar{y})$ is also an equivalence formula over A . The reader can also prove that equivalence formulas are closed under existential quantification, that is, if $\delta(\bar{x} \bar{y})$ is an equivalence formula over A then $\exists \bar{y} \delta(\bar{x} \bar{y})$ is an equivalence formula over A (hint: observe that you can substitute the existential quantification with a finite disjunction). EXERCISE

12.3.5 Cosets as parameters.

It is useful to represent equivalence formulas from a different perspective: the intuition behind the following proposition is that if $\delta(\bar{b} \bar{u})$ does not really depends on \bar{b} but only on its equivalence class, then $\delta(\bar{b} \bar{x})$ is an equivalence formula.

PROPOSITION Let \mathcal{D} be a definable set. The following are equivalent:

- 1 \mathcal{D} is defined by an equivalence A -formula; and
- 2 there are a A -formula $\delta(\bar{z} \bar{x})$ and some finite equivalence relation $\varepsilon(\bar{z}, \bar{w})$ over A such that $\delta(\bar{b} \bar{x})$ defines \mathcal{D} for every \bar{b} in some coset of $\varepsilon(\bar{z}, \bar{w})$.

PROOF The implication 1 \Rightarrow 2 is clear if we write $\delta(\bar{b} \bar{x})$ in a form as in (\diamond) of Paragraph 12.3.2 above. To prove the implication 2 \Rightarrow 1 we can assume that $\delta(\bar{b}_1 \bar{x})$ is equivalent to $\delta(\bar{b}_2 \bar{x})$ whenever $\varepsilon(\bar{b}_1, \bar{b}_2)$. If not, then we simply replace $\delta(\bar{z} \bar{x})$ with the formula $\exists \bar{w} [\varepsilon(\bar{z}, \bar{w}) \wedge \delta(\bar{w} \bar{x})]$. Now define the equivalence relation

$$* \quad \eta(\bar{x}, \bar{y}) := \forall \bar{z} [\delta(\bar{z} \bar{x}) \leftrightarrow \delta(\bar{z} \bar{y})].$$

If we prove that this is a finite equivalence relation then, since obviously

$$\forall \bar{x} \bar{y} [\eta(\bar{x}, \bar{y}) \rightarrow [\delta(\bar{b} \bar{x}) \leftrightarrow \delta(\bar{b} \bar{y})]],$$

we obtain that $\delta(\bar{b} \bar{x})$ is an equivalence formula over A as required. So suppose for a contradiction that there is an infinite set of tuples such that $\neg \eta(\bar{c}_i, \bar{c}_j)$ for every $i < j \in \omega$. Let $\bar{z}_{i,j}$ witness the failure of the universal quantifier in the definition (*) above. Then clearly for every $i < j < k$ it cannot be that both $\varepsilon(\bar{z}_{i,j}, \bar{z}_{i,k})$ and $\varepsilon(\bar{z}_{i,j}, \bar{z}_{j,k})$. A contradiction is obtained with the following Ramsey argument. Think the $\varepsilon(\bar{z}, \bar{w})$ -equivalence classes as colors. Color each unordered pair $\{i, j\} \subseteq \omega$ with the class of $\bar{z}_{i,j}$. By the Ramsey theorem there is an homogeneous set containing some elements $i < j < k$.

12.3.6 A syntactic characterization of almost-definability.

Using the characterization above, we prove now the important fact announced above: sets almost-over A are exactly those definable by equivalence A -formulas.

PROPOSITION Let \mathcal{D} be a definable set and let A be a set of parameters. The following are equivalent:

- 1 \mathcal{D} is almost-over A ; and
- 2 \mathcal{D} is definable by an equivalence formula over A .

PROOF We prove the non-trivial direction using the characterization of equivalence formulas claimed above. Fix a parameter-free formula $\delta(\bar{x} \bar{z})$ and a \bar{b} such that $\delta(\bar{b} \bar{z})$ defines \mathcal{D} . Define the following abbreviation

$$\varepsilon(\bar{x}, \bar{y}) := \forall \bar{z} [\delta(\bar{x} \bar{z}) \leftrightarrow \delta(\bar{y} \bar{z})].$$

Let $p(\bar{x})$ be the A -type of \bar{b} . The A -orbit of \mathcal{D} is finite, say, it has cardinality n . So there are $\bar{b}_1 \cdots \bar{b}_n$ such that

$$p(\bar{x}) + \bigwedge_{i=1}^n \neg \varepsilon(\bar{b}_i \bar{x})$$

is inconsistent. By saturation, for some $\xi(\bar{x})$ in $p(\bar{x})$

$$\xi(\bar{x}) \rightarrow \bigvee_{i=1}^n \varepsilon(\bar{b}_i \bar{x}).$$

It is immediate to check that the formula

$$\eta(\bar{x}, \bar{y}) := [\xi(\bar{x}) \wedge \xi(\bar{y}) \wedge \varepsilon(\bar{x}, \bar{y})] \vee [\neg \xi(\bar{x}) \wedge \neg \xi(\bar{y})]$$

defines an equivalence relation with exactly $n+1$ equivalence classes. It is clear that $\delta(\bar{a} \bar{x})$ is equivalent to $\delta(\bar{b} \bar{x})$ for every \bar{a} such that $\eta(\bar{b}, \bar{a})$.

12.3.7 Almost-almost-over = almost-over.

It may seem natural to consider almost-almost-definability, but nothing new appears. The reader can prove the following as an exercise. Let $\varepsilon(\bar{x} \bar{y})$ be a finite equivalence relation definable almost-over A . Then all cosets of $\varepsilon(\bar{x} \bar{y})$ are definable almost-over

A . (This is a second-order version of $\text{acl}(\text{acl } A) = \text{acl } A$, in a sense that we will make precise in the next chapter.)

EXERCISE

12.3.8 An exercise.

Prove that if $\delta(x)$ is almost-over A then there is an A -formula $\psi(x)$ such that $\delta(A) = \psi(A)$.

EXERCISE

Chapter 13

Elimination of imaginaries

In this chapter we show that definable sets may play the role of parameters. In the first section we discuss this and in the second section we introduce elimination of imaginaries: a phenomenon that collapses the second- to the first-order. In the last section we see how elimination of imaginaries can be obtained via a syntactic trick.

Notation and global assumptions.

As in Chapter 6 above.

13.1 Definable sets as parameters.

The similarity between elements and definable sets can be pushed further. Definable sets generalize elements also in the role of parameters. Most of the results encountered so far remain valid if we allow the set of parameters (usually denoted by A) to contain also definable sets. When they occur in this context definable sets are called *imaginary elements*. In this section we will give a sense to terms like: *A-formula*, *A-type*, *A-automorphism*, etc. when A contains imaginaries. In this section we use a second-order formalism. This has some technical nuisances, but I find it is the most direct way to introduce the relevant ideas. In the last section we present a many-sorted formalism: the so called *eq* expansion (which is the ‘official way’ of dealing with imaginaries).

13.1.1 Expansions of the language.

Let A be the union of two sets of small cardinality A_{1st} and A_{2nd} . The set A_{1st} contains just elements of the universe, that is, the usual parameters, these are now called, for emphasis, *first-order* or *real* parameters. The set A_{2nd} contains definable subsets of the universe; these are the *second-order* or *imaginary* elements of A .

We define A formulas as follows: every A_{1st} -formula and every formula defining a set in A_{2nd} is an A -formula. Further A -formulas are closed under all logical connectives. In other words, A -formulas contain arbitrary parameters but the parameters not in A_{1st} may occur only inside a subformula that defines an imaginary element of A .

One may wonder if the introduction of second-order parameters really brings something new into the scene. After all, every A -formula is equivalent to a B -formula for some set B containing only first-order parameters: B need only contain A_{1st} and define all sets in A_{2nd} . But note that there may be no set B of first-order parameters such that B -definable sets are *exactly* the A -definable sets.

An A -type is a consistent set of A -formulas. The number of A -formulas may not be small, this because there may be κ formulas all defining the very same set. But if we count formulas up to equivalence, then there are at most $|TA|$ formulas over A . So every finitely consistent set of A formulas is a type.

An A -automorphism is an automorphism that fixes point wise A_{1st} and fixes setwise all elements of A_{2nd} .

13.1.2 Definability and almost-definability revisited.

For easy of reference we restate the main results proved in the last two chapters. Now these facts are read with an extended notion of parameter. The reader can easily check that the proofs above actually prove the following claims.

PROPOSITION Let A be a set of real and/or imaginary parameters. Let \mathcal{D} be a definable set. The following are equivalent:

- 1 \mathcal{D} is definable over A ;
- 2 if $\bar{a} \equiv_A \bar{b}$ then $\bar{a} \in \mathcal{D}$ if and only if $\bar{b} \in \mathcal{D}$; and
- 3 every A -automorphism fixes \mathcal{D} .

Moreover, the following are equivalent:

- 4 \mathcal{D} is almost-over A ;
- 5 the orbit of A under A -automorphisms has small cardinality;
- 6 the orbit of \mathcal{D} under A -automorphisms is finite; and
- 7 \mathcal{D} is definable by an equivalence A -formula.

13.1.3 Algebraic closures of second-order parameters.

Let A be a set of first- and second-order parameters. We say that b is algebraic over A if b is a solution of some algebraic A -formula. It is easy to see that b is algebraic over A if and only if b belongs to every model defining all elements of A . In particular, for every formula $\varphi(\bar{x})$, the intersection of all models defining $\varphi(\bar{U})$ coincides with $\text{acl}\{\varphi(\bar{U})\}$, the algebraic closure of $\{\varphi(\bar{U})\}$.

13.2 Elimination of imaginaries

A theory has elimination of imaginaries if every definable set is interdefinable with a tuple of elements of the universe. This is a phenomenon that can considerably simplify our work: each time a definable set is needed in the role of a parameter we can replace it with the tuple this set is interdefinable with. Interdefinability (definitions can be inverted) guarantees that the substitution cause no loss of information.

13.2.1 Elimination of imaginaries.

We say that T has **elimination of imaginaries** if for every definable set \mathcal{D} there is a parameter-free formula $\psi(\bar{x}\bar{z})$ such that

$$(*) \quad \exists^1 \bar{z} \forall \bar{x} \left[\bar{x} \in \mathcal{D} \leftrightarrow \psi(\bar{x}\bar{z}) \right],$$

We include the case when the arity of \bar{z} may be 0: then we agree that $\exists^1 \bar{z}$ is empty so we read $(*)$ as saying that \mathcal{D} is definable by a parameter-free formula. We call the witness of $\exists \bar{z}$ above a **canonical parameter of \mathcal{D}** , informally one calls it also a *name* or a *code* of the definable set. So $(*)$ claims that, modulo the formula $\psi(\bar{x}\bar{z})$, this names are not ambiguous. Observe that we cannot say *the* canonical parameter of \mathcal{D} because we may have different canonical parameters for different formulas $\psi(\bar{z}\bar{x})$. But note that any two canonical parameters of \mathcal{D} are interdefinable.

13.2.2 Weak elimination of imaginaries.

We say that T has **weak elimination of imaginaries** if the property $(*)$ above holds with $\exists^1 \bar{z}$ for $\exists^n \bar{z}$ (where n is positive and depends on \mathcal{D}). In this case a finite set of tuples uniquely determine the set \mathcal{D} . Also in this case \bar{z} may be empty.

13.2.3 A remark.

The property required for elimination of imaginaries refers essentially to the universe. As an exercise the reader can verify that this reference is immaterial (it does not matter in which universe we test the property). So elimination of imaginaries is indeed a property of T . EXERCISE

13.2.4 Canonical parameters and interdefinability.

The following is a more intrinsic characterization canonical parameters. We say that **\bar{a} and \mathcal{D} are interdefinable** if \mathcal{D} is definable over \bar{a} , and \bar{a} is definable over $\{\mathcal{D}\}$.

PROPOSITION Let \mathcal{D} be a definable set. The following are equivalent:

- 1 \bar{a} is a canonical parameter of \mathcal{D} ;
- 2 \bar{a} and \mathcal{D} are interdefinable; and

3 every automorphism that fixes \bar{a} fixes \mathcal{D} setwise and vice versa.

PROOF The equivalence $2 \Leftrightarrow 3$ is Proposition 13.1.2 and the implication $1 \Rightarrow 2$ is clear from $(*)$ above. To prove $2 \Rightarrow 1$ let \bar{a} be interdefinable with \mathcal{D} . Let $\varphi(\bar{z}\bar{x})$ be a parameter-free formula such that $\varphi(\bar{a}\bar{x})$ defines \mathcal{D} . Observe that if $\bar{b} \equiv \bar{a}$ and $\varphi(\bar{b}\bar{x})$ defines \mathcal{D} then $\bar{b} = \bar{a}$ (in fact, it follows that $\bar{b} \equiv_{\{\mathcal{D}\}} \bar{a}$ but \bar{a} is definable over $\{\mathcal{D}\}$). So, writing $p(\bar{z})$ for the parameter-free type of \bar{a} , the type

$$p(\bar{z}) + \forall \bar{x} \left[\varphi(\bar{a}\bar{x}) \leftrightarrow \varphi(\bar{z}\bar{x}) \right]$$

has only one realization. By saturation, there is a formula $\psi(\bar{z})$ in $p(\bar{z})$ such that

$$\exists^1 \bar{z} \left[\psi(\bar{z}) \wedge \forall \bar{x} \left[\varphi(\bar{a}\bar{x}) \leftrightarrow \varphi(\bar{z}\bar{x}) \right] \right].$$

By logic, we can write

$$\exists^1 \bar{z} \forall \bar{x} \left[\varphi(\bar{a}\bar{x}) \leftrightarrow \left[\psi(\bar{z}) \wedge \varphi(\bar{z}\bar{x}) \right] \right]$$

and, since \bar{a} is trivially a witness of $\exists^1 \bar{z}$ above, we obtain 1 as wanted.

13.2.5 A remark

Assume elimination of imaginaries. Let A be a set of first- and second-order parameters. Let B be $\text{dcl } A$, the set those elements of the universe that are definable by A -formulas. Then the notions of A -formula, A -type, A -automorphism, etc. introduced in Section 13.1 above coincide, respectively, with the notions of B -formula, B -type, B -automorphism, etc., moreover a model M defines all elements of A if and only if M contains B .

13.2.6 Weak elimination of imaginaries and interdefinability.

We say that two definable sets \mathcal{D} and \mathcal{C} are **interdefinable** if some formula over $\{\mathcal{C}\}$ defines \mathcal{D} and, vice versa, some formula over $\{\mathcal{D}\}$ defines \mathcal{C} .

PROPOSITION The following are equivalent:

- 1 T has weak elimination of imaginaries;
- 2 for every definable set \mathcal{D} there is a finite set A interdefinable with \mathcal{D} ; and
- 3 for every definable set \mathcal{D} there is a finite set A such that every automorphism of that fixes A setwise fixes \mathcal{D} setwise and vice versa.

PROOF As above: the equivalence $2 \Leftrightarrow 3$ is a consequence Proposition 13.1.2 and the implication $1 \Rightarrow 2$ is clear from Definition 13.2.2 above. To prove $2 \Rightarrow 1$ let \mathcal{D} be an arbitrary definable set and let A be interdefinable with \mathcal{D} . So there is a parameter-free formula $\varphi(\bar{z}\bar{x})$ such that $\varphi(\bar{a}\bar{x})$ defines \mathcal{D} for every tuple \bar{a} that is the concatenation of all the tuples in A . Let F be an elementary map and let \bar{b} be concatenation of the tuples in FA . We claim that if $\varphi(\bar{b}\bar{x})$ defines \mathcal{D} then FA is just a permutation of A . In fact, if $\varphi(\bar{b}\bar{x})$ defines \mathcal{D} then F fixes \mathcal{D} setwise. Since A is definable over $\{\mathcal{D}\}$ then F fixes also A . So, writing $p(\bar{z})$ for the parameter-free

type of \bar{a} , we have that all tuple realizing the type

$$p(\bar{z}) + \forall \bar{x} \left[\varphi(\bar{z} \bar{x}) \leftrightarrow \varphi(\bar{a} \bar{x}) \right]$$

are permutations of each other. By saturation, there is a formula $\psi(\bar{z})$ in $p(\bar{z})$ such that all solutions of

$$\psi(\bar{z}) \wedge \forall \bar{x} \left[\varphi(\bar{z} \bar{x}) \leftrightarrow \varphi(\bar{a} \bar{x}) \right].$$

are a permutation of each other. So, for some n ,

$$\exists =^n \bar{z} \forall \bar{x} \left[\left[\psi(\bar{z}) \wedge \varphi(\bar{z} \bar{x}) \right] \leftrightarrow \varphi(\bar{a} \bar{x}) \right]$$

as wanted.

13.2.7 Weak elimination of imaginaries \rightarrow almost-over $A =$ over $\text{acl } A$.

Now we show how elimination of imaginaries simplifies the theory of definability in the non-stationary case.

PROPOSITION Let T have weak elimination of imaginaries. Let \mathcal{D} be an arbitrary definable set. The following are equivalent:

1 \mathcal{D} is almost-over A ; and

2 \mathcal{D} is over $\text{acl } A$.

PROOF The direction 2 \Rightarrow 1 is obvious. To prove 1 \Rightarrow 2 assume \mathcal{D} is almost-over A . Let $\psi(\bar{x} \bar{z})$ be such that

$$* \quad \exists =^k \bar{z} \forall \bar{x} \left[\bar{x} \in \mathcal{D} \leftrightarrow \psi(\bar{x} \bar{z}) \right].$$

Let \bar{b}_i for $i = 1 \dots k$ be the witnesses of the existential quantifier above. By 7.1.6 above, it suffices to show that these witnesses belong to every model M containing A . But this is clear: if $\vartheta(\bar{x})$ be an M -formula that defines \mathcal{D} then substituting $\vartheta(\bar{x})$ for $\bar{x} \in \mathcal{D}$ we obtain a M -formula. By elementarity all witnesses of $\exists =^k \bar{z}$ are in M .

13.2.8 Algebraic witnesses \rightarrow weak elimination of imaginaries.

Below we prove that the theory of algebraically closed fields and the theory of real closed fields have elimination of imaginaries. Both results use the following lemma.

PROPOSITION If T has algebraic witnesses then T has weak elimination of imaginaries.

PROOF Assume the hypotheses. By the remark in 13.1.3 above, every consistent formula $\varphi(\bar{x})$ has a solution that is algebraic over $\{\varphi(\bar{u})\}$. Now let \mathcal{D} be the set $\varphi(\bar{u} \bar{a})$ where $\varphi(\bar{x} \bar{z})$ is a parameter-free formula and \bar{a} is arbitrary. Consider the formula

$$\xi(\bar{z}) = \forall \bar{x} \left[\bar{x} \in \mathcal{D} \leftrightarrow \varphi(\bar{x} \bar{z}) \right].$$

By the claim, $\xi(\bar{z})$ has a solution that is algebraic over $\{\xi(\bar{u})\}$, hence over $\{\mathcal{D}\}$. Fix any finite set A definable over $\{\mathcal{D}\}$ that is subset of $\xi(\bar{u})$. As any element of A

defines \mathcal{D} , then \mathcal{D} is interdefinable with A . Then weak elimination of imaginaries follows from Proposition 13.2.6 above.

13.2.9 Weak elimination of imaginaries in strongly minimal theories

The following corollary is worth of mention.

PROPOSITION If T is strongly minimal and $\text{acl } \emptyset$ is infinite then T has weak elimination of imaginaries.

PROOF Observe that, by Exercise 8.1.18 above, every strongly minimal theory has algebraic witnesses as soon as $\text{acl } \emptyset$ is infinite.

13.2.10 Algebraically closed fields have elimination of imaginaries.

The theory of the algebraically closed fields (of any characteristic) is strongly minimal, moreover $\text{acl } \emptyset$ is infinite, so the lemma above apply. With a little of algebra we obtain more.

PROPOSITION Let T be the theory of algebraically closed field of characteristic p , where p is arbitrary. Then T has elimination of imaginaries.

PROOF Since T is strongly minimal and $\text{acl } \emptyset$ is infinite, then T has algebraic witnesses and, by the lemma above, it has weak elimination of imaginaries. So it suffices to show that for every finite set of tuples A is interdefinable with a tuple \bar{c} . We denote by $a_{i,1}, \dots, a_{i,m}$ the components of the tuple \bar{a}_i . Let \bar{c} be a tuple that lists the coefficients of the polynomial

$$\prod_{i=1}^n \left(y + \sum_{j=1}^m x_j a_{i,j} \right)$$

This tuple is definable over $\{A\}$ and, since in T there is a unique factorization, A is definable over \bar{c} .

13.2.11 Real closed fields have elimination of imaginaries.

The theory of real closed fields has been defined in Section 5.3 above. There we also prove elimination of quantifiers.

PROPOSITION Let T be the theory of real closed fields. Then T has elimination of imaginaries.

PROOF Since T defines a linear order, elimination of imaginaries is equivalent to weak elimination. We show that T has algebraic witnesses and the proposition follows from the lemma above. By Exercise 13.1.3 above, it suffices to show that every consistent formula $\varphi(x)$ has a solution definable over $\{\varphi(\mathcal{U})\}$. By elimination of quantifiers, $\varphi(\mathcal{U})$ is union of finitely many disjoint intervals, that is, sets of the form (a, b) , $(a, +\infty)$, $(-\infty, a)$, or $\{a\}$. So there exist a least element a such that $(a, +\infty)$

is either subset of $\varphi(\mathcal{U})$ or it is disjoint of $\varphi(\mathcal{U})$. Since a is definable over $\{\varphi(\mathcal{U})\}$ we are done if at least one of a , $a + 1$, or $a - 1$ satisfy to $\varphi(x)$. If all these three cases fail, then (since $\varphi(\bar{x})$ is consistent) there is a least b such that (b, a) is a subset of $\varphi(\mathcal{U})$. Then $(a + b)/2$ is an solution of $\varphi(x)$ definable over $\{\varphi(\mathcal{U})\}$.

13.2.12 Uniform elimination of imaginaries.

Let Φ be a family of definable sets. We say that the sets in Φ are **uniformly definable** or that the Φ is a **uniform family of definable sets**, if there is a formula $\varphi(\bar{x} \bar{z})$ such that

$$\Phi = \left\{ \varphi(\bar{\mathcal{U}} \bar{a}) : \bar{a} \subseteq \mathcal{U} \right\}.$$

Elimination of imaginaries as defined in 13.2.2 is not the most convenient notion to deal with uniform families of definable sets: for each set in Φ condition (*) could obtain with a different formula $\psi(\bar{x} \bar{z})$. This is the reason why we introduce the following apparently stronger notion. We say that **T has uniform elimination of imaginaries** if for every parameter-free formula $\varphi(\bar{x} \bar{y})$ there is a parameter-free formula $\psi(\bar{x} \bar{z})$ such that

$$\forall \bar{y} \exists^1 \bar{z} \forall \bar{x} \left[\varphi(\bar{x} \bar{y}) \leftrightarrow \psi(\bar{x} \bar{z}) \right].$$

To appreciate the notion consider the following example: let \mathcal{U} be a group and let \mathcal{G} be a definable subgroup of \mathcal{U} , then the group \mathcal{U}/\mathcal{G} is a uniform family of definable sets. In this case, uniform elimination of imaginaries yields the existence of a definable subgroup of \mathcal{U} isomorphic to \mathcal{U}/\mathcal{G} .

13.2.13 Definable equivalence relations.

Often uniform elimination of imaginaries is described using \emptyset -definable equivalence relations. To every formula $\varphi(\bar{x} \bar{y})$ we associate the equivalence relation

$$\varepsilon(\bar{a} \bar{b}) := \forall \bar{x} \left[\varphi(\bar{x} \bar{a}) \leftrightarrow \varphi(\bar{x} \bar{b}) \right]$$

that says that \bar{a} and \bar{b} define the same set. If $\varphi(\bar{x} \bar{z})$ is parameter-free then $\varepsilon(\bar{w} \bar{z})$ is parameter-free. It is easy to check that the existence of a formula $\psi(\bar{x} \bar{z})$ as required in Paragraph 13.2.2 is equivalent to the existence of a parameter-free formula $\vartheta(\bar{w} \bar{z})$ such that

$$\forall \bar{u} \exists^1 \bar{z} \forall \bar{w} \left[\varepsilon(\bar{w} \bar{u}) \leftrightarrow \vartheta(\bar{w} \bar{z}) \right]$$

That is, the formula $\vartheta(\bar{w} \bar{z})$ defines the graph of a total function $F(\bar{w}) = \bar{z}$ that is constant exactly on each equivalence class of $\varepsilon(\bar{w} \bar{u})$.

13.2.14 Uniformity comes (practically) for free.

Now we show that, under very weak hypothesis, uniform elimination of imaginaries follows from elimination of quantifiers.

PROPOSITION Suppose that in the universe there are (at least) two \emptyset -definable elements. Then the following are equivalent:

- 1 T has elimination of imaginaries; and
- 2 T has uniform elimination of imaginaries.

PROOF One direction is clearly trivial. So, assume that T has elimination of imaginaries and suppose for a contradiction that this is not uniform. There is some parameter-free formula $\varphi(\bar{x} \bar{y})$ such that for every parameter-free formula $\psi(\bar{x} \bar{z})$ the following formula is consistent

$$\# \quad \neg \exists^1 \bar{z} \forall \bar{x} \left[\varphi(\bar{x} \bar{y}) \leftrightarrow \psi(\bar{x} \bar{z}) \right].$$

For reasons that will be clear below we need to assume that for every \bar{a} the set $\varphi(\bar{\mathcal{U}} \bar{a})$ is non-empty. This is not a loss of generality: if not then replace $\varphi(\bar{x} \bar{y})$ with the formula

$$\varphi^*(v \bar{x} \bar{y}) := \left[\neg \exists \bar{x} \varphi(\bar{x} \bar{y}) \wedge v = c \right] \vee \left[\exists \bar{x} \varphi(\bar{x} \bar{y}) \wedge v = d \wedge \varphi(\bar{x} \bar{y}) \right]$$

where we c, d are distinct \emptyset -definable elements. Clearly $\forall \bar{y} \exists v \bar{x} \varphi^*(v \bar{x} \bar{y})$ and, for the claim to be proved, the two formulas $\varphi^*(v \bar{x} \bar{y})$ and $\varphi(\bar{x} \bar{y})$ are perfectly equivalent: in fact for every \bar{a} the two sets $\varphi(\bar{\mathcal{U}} \bar{a})$ and $\varphi^*(\mathcal{U} \bar{\mathcal{U}} \bar{a})$ are interdefinable. Now, let $p(\bar{y})$ be the collection of all formulas of the form (b) above as $\psi(\bar{x} \bar{z})$ ranges over the parameter-free formulas. Clearly $p(\bar{y})$ cannot be consistent: if \bar{a} is a realization of $p(\bar{y})$ then $\varphi(\bar{x} \bar{a})$ contradicts elimination of imaginaries. So for some finite number of parameter-free formulas $\psi_1(\bar{x} \bar{z}) \dots \psi_n(\bar{x} \bar{z})$ such that

$$\# \quad \bigvee_{i=1}^n \exists^1 \bar{z} \forall \bar{x} \left[\varphi(\bar{x} \bar{y}) \leftrightarrow \psi_i(\bar{x} \bar{z}) \right].$$

Let $\bar{c}_1 \dots \bar{c}_n$ be distinct definable tuples of the same arity (it is immediate that with two definable elements one can obtain an arbitrary finite number of definable tuples). We would like to conclude that

$$\exists^1 \bar{z} \bar{w} \forall \bar{x} \left[\varphi(\bar{x} \bar{y}) \leftrightarrow \bigvee_{i=1}^n \left[\psi_i(\bar{x} \bar{z}) \wedge \bar{w} = \bar{c}_i \right] \right].$$

Clearly this is the case if the disjunction in (#) never holds for more than one disjunct simultaneously. If not then we replace $\psi_i(\bar{x} \bar{z})$ with

$$\psi_i^\#(\bar{x} \bar{z}) := \psi_i(\bar{x} \bar{z}) \wedge \bigwedge_{j < i} \neg \exists \bar{w} \forall \bar{x} \left[\psi_j(\bar{x} \bar{w}) \leftrightarrow \psi_i(\bar{x} \bar{z}) \right].$$

Clearly this replacement does not affect the validity of (#). Note that the effect of the replacement is of substituting an inconsistent formula for $\psi_i(\bar{x} \bar{z})$ whenever $\psi_j(\bar{x} \bar{w})$ defines the same set for some \bar{w} and some $j < i$. So the only way two disjuncts of (#) can hold simultaneously is when $\neg \exists \bar{x} \varphi(\bar{x} \bar{y})$ which has been excluded by the hypothesis above.

13.3 The eq-theory

There is a simple syntactic trick that allows to work with any theory as it was a theory with elimination of *imaginaries*: add new elements to the domain, the names of the definable sets. We need to keep these new elements distinct from the original ones (as we want to leave essentially unchanged the metaproperties of T), so the new elements will be given a different *sort*. Unfortunately (for a reason that we comment below) we must work with a kind of language that is not properly first-order: a *many-sorted language*.

13.3.1 Many-sorted languages.

We introduce many-sorted languages in some generality. A many-sorted language consists of

- 1 the disjoint union of three sets: the set of functions, set of relations and a collection of objects that we call **sorts**; and
- 2 a map that assigns to every relation r of L a tuple of sorts $\langle s_1 \cdots s_n \rangle$ which we call improperly **the sort of r** , to every function f a tuple of sorts of positive arity $\langle s_1 \cdots s_{n+1} \rangle$ which we call improperly **the sort of f** . The the last element s_{n+1} of this tuple is called the **target sort of f** the renaming elements $\langle s_1 \cdots s_{n+1} \rangle$ are called the **input sort of f** . When n is 0 we call f a constant.

13.3.2 Many-sorted structures.

A many-sorted structure M of signature L consists of

- 1 a map that assigns to each sort s a set M_s called the interpretation of s ;
- 2 for every relation r of L a subset r^M of $M_{s_1} \times \cdots \times M_{s_n}$ where $\langle s_1 \cdots s_n \rangle$ is the sort of r ; and
- 3 for every function f of L a function f^M from $M_{s_1} \times \cdots \times M_{s_n}$ to $M_{s_{n+1}}$ where $\langle s_1 \cdots s_{n+1} \rangle$ is the sort of f .

13.3.3 A remark.

There is an easy way to simulate a many-sorted language with a first-order language. To every many-sorted language L_m we associate a first-order language L_1 that contains the same symbols for relations and functions as L_m and a unary predicate U_s for each sort s . Let us call a structures M of signature L_1 *good* if:

- 1 the interpretation of the predicates U_s forms partition of the domain of the structure;
- 2 every relation r^M holds only for tuples that have the same sort as r ; and
- 3 every functions f^M outputs elements with the target sort of f .

Given a good structure we can obtain a many-sorted structure in an obvious way. The converse is also true (only, the procedure is not unique, we can decide arbitrarily the value of $f^M(\bar{a})$ when tuple \bar{a} is not of the input source of f). The drawback of this formalization is that, as soon as L_m contains infinitely, no saturated model can satisfy (1). This undesirable feature makes prefer the many-sorted approach.

13.3.4 Many-sorted formulas.

Many-sorted terms are defined along the same lines as one-sorted terms but now each term receives a sort. To begin with, for each sort s , we fix an infinite set of variables: the variables of sort s . Variables are atomic terms, parameters are also atomic terms; each comes its obvious sort. If $\langle t_1 \cdots t_n \rangle$ is a tuple of terms of sort $\langle s_1 \cdots s_n \rangle$ and f is a function of sort $\langle s_1 \cdots s_{n+1} \rangle$ then $f(t_1 \cdots t_n)$ is a term of sort s_{n+1} . Nothing else is a term. Atomic formula have the form $r(t_1 \cdots t_n)$ where the sort $\langle s_1 \cdots s_n \rangle$ of $\langle t_1 \cdots t_n \rangle$ coincide with the sort of r . The rest of the definition follows that of Chapter 2 and may be entrusted to the reader (note that quantifiers also need receive a sort). The reader should also convince him/herself that all basic model theory goes trough in this situation, in particular the compactness theorem and the existence of saturated models. It is important to note that since every variable comes with its sort we cannot use compactness to enforce the existence of elements with no sort.

13.3.5 The eq-language.

Fix first-order language L and a complete theory T . The language L^{eq} is a many sorted language, its definition depends on T . It contains a sort s for every parameter-free formula $\varepsilon(\bar{x}, \bar{y})$ of L that defines (according to T) an equivalence relation. In particular there is a sort for the relation $x = y$: this we call the **fundamental sort** and denote it by s_* . All functions and relations of L are in L^{eq} . The tuple of sorts associated with the relation r of L is $\langle s_*, \dots, s_* \rangle$, where the arity of this tuple is that r . The tuple of sorts associated with the function f of L is $\langle s_*, \dots, s_* \rangle$ where the arity of this tuple is that of f plus one. In L^{eq} there is also a function symbol f_s for every sort s . The input sort of f_s is $\langle s_*, \dots, s_* \rangle$ and has the same arity as \bar{x} and/or \bar{y} in the relation $\varepsilon(\bar{x}, \bar{y})$ corresponds to the sort s . The target sort of f_s is s . Note that there is an obvious way of reading formulas of L as formulas of L^{eq} .

13.3.6 The eq-models.

Every model M can be expanded in a canonical way to a many-sorted structure of signature L^{eq} . The expanded structure is denoted by M^{eq} . The interpretation of the fundamental sort is M itself. (So M and M^{eq} agree on the evaluation of sentences of L .) The interpretation M_s of the sort s that corresponds to the relation $\varepsilon(\bar{x}, \bar{y})$

is the set of equivalence classes that this relation defines in \bar{M} . The functions and relations of L^{eq} that are in L are interpreted in the usual way. The function f_s maps each tuple of M to its equivalence class.

13.3.7 The eq-theory.

The theory T^{eq} consists of all the sentences of T together with axioms which assert that $f_s(\bar{x}) = f_s(\bar{y})$ if and only if $\varepsilon(\bar{x}, \bar{y})$ (where $\varepsilon(\bar{x}, \bar{y})$ is the relation which correspond to the sort s and \bar{x}, \bar{y} are variables of the fundamental sort). There are also axioms that say that f_s is a surjective map (between the input and the target sorts). Without loss of generality we can assume that every structure of signature L^{eq} that models T^{eq} is obtained from a model of T with the procedure explained above. The reader can also check that M^{eq} is saturated if and only if M is saturated (many other properties of M are inherited by M^{eq}).

13.3.8 A lemma.

We list without proof some basic facts about T^{eq} and \mathcal{U}^{eq} . Note especially the second claim below: this tells that we can pin down definable subsets of \mathcal{U}^{eq} to definable subsets of \mathcal{U} .

PROPOSITION We identify \mathcal{U} with a subset of \mathcal{U}^{eq} . For every model M we identify M^{eq} with a substructure of \mathcal{U}^{eq} .

- 1 the theory T^{eq} is complete and without finite models, \mathcal{U}^{eq} is a saturated model of T^{eq} ;
- 2 for every parameter-free formula $\varphi(x_1 \cdots x_n)$ of L^{eq} , where $x_1 \cdots x_n$ have sort $s_1 \cdots s_n$, there is a parameter-free formula $\psi(\bar{y}_1 \cdots \bar{y}_n)$ of L that is equivalent to $\varphi(f_{s_1}(\bar{y}_1) \cdots f_{s_n}(\bar{y}_n))$ for $\bar{y}_1 \cdots \bar{y}_n$ ranging over \mathcal{U} ;
- 3 M^{eq} is an elementary substructure of \mathcal{U}^{eq} for every model M ;
- 4 every element of \mathcal{U}^{eq} is definable over \mathcal{U} ;
- 5 every subset of $\bar{\mathcal{U}}$ which is definable over \mathcal{U}^{eq} is definable over \mathcal{U} ;
- 6 \mathcal{U}^{eq} is saturated and has the same cardinality as \mathcal{U} .

13.3.9 Elimination of imaginaries.

Finally we obtain the result we were aiming for.

PROPOSITION The theory T^{eq} has elimination of imaginaries.

PROOF Let $\varepsilon(x_1 \cdots x_n, y_1 \cdots y_n)$ be parameter-free formula of L^{eq} that defines an equivalence relation in \mathcal{U}^{eq} among tuples of sort $\langle s_1 \cdots s_n \rangle$. By the second claim in the proposition above there is a parameter-free formula $\varphi(\bar{x}_1 \cdots \bar{x}_n, \bar{y}_1 \cdots \bar{y}_n)$ of L that is equivalent to

$$\varepsilon(f_{s_1}(\bar{x}_1) \cdots f_{s_n}(\bar{x}_n), f_{s_1}(\bar{y}_1) \cdots f_{s_n}(\bar{y}_n)).$$

So $\varphi(\bar{x}_1 \cdots \bar{x}_n, \bar{y}_1 \cdots \bar{y}_n)$ defines an equivalence relation among tuples of the fundamental sort. Let s be the sort associated to this equivalence relation and let f_s be the function that gives the equivalence class of the tuple $\bar{c}_1 \cdots \bar{c}_n$. Recall that $f_{s_1} \cdots f_{s_n}$ are surjective, that is, for every a of sort s_i there is a tuple \bar{c} of the fundamental sort such that $f_{s_i}(\bar{c}) = a$. The formula

$$\exists \bar{y}_1 \cdots \bar{y}_n \left[f_s(\bar{y}_1 \cdots \bar{y}_n) = z \wedge \bigwedge_{i=1}^n f_{s_i}(\bar{y}_i) = x_i \right]$$

defines a one-to-one correspondence between equivalence classes of $\varepsilon(x_1 \cdots x_n, y_1 \cdots y_n)$ and equivalence classes of $\varphi(\bar{x}_1 \cdots \bar{x}_n, \bar{y}_1 \cdots \bar{y}_n)$. The latter are elements of \mathcal{U}^{eq} .

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Index

$M \models p(\bar{a})$, 20
 $M \models \varphi$, 18
 $M \models \varphi(\bar{x})$, 19
 $M \preceq_{\text{qf}} N$, 7
 \perp , 15
 \vdash , 36
 $\text{tp}(\bar{a}/A)$, 62
 \bar{a}, \bar{c} , etc. \bar{a}, \bar{c} , etc., 7
 $\text{rng } \bar{a}, \text{rng } \bar{a}$, 7
 $A + \bar{a}$, 7
 AB , 7
 $\langle A \rangle_M, \langle \bar{a} \rangle_M$, 8
 $\text{acl } A$, 74
 $\text{dcl } A$, 74
 $\text{acl}\{\mathcal{D}\}$, 140
 $\text{acl}\{\varphi(\bar{U})\}$, 140
 $\text{acl } A$, 140
 $\sqrt{\exists}T, \exists T, \forall\exists T, \exists\sqrt{\forall}T$, 52
 $_{\text{qf}}T, \text{qf}p(\bar{x})$, 36
 $\text{Th}_A M, \text{Th}M$, 23
 $_{\text{qf}}\text{Th}_A M, \text{qf}\text{Th}M$, 23
 $F : M \rightarrow N$, 8
 $\text{rng } F$, 8
 $\text{dom } F$, 8
 $F\varphi(\bar{x}), Fp(\bar{x})$, 29
 $t(\bar{x})$, 14
 $\varphi(\bar{x})$, 16
 $\varphi(\bar{x}), t(\bar{x})$, 17

A

algebraic
 T has – witnesses, 78
 closure, 74
 element, 73
 formula, 73
algebraically closed, 74

almost
 –definable, 132
 –in, 74
 –over, 132
 –satisfied, 79
amalgamation, 122
 elementary – lemma, 124
 free, 124
 quantifier-free – lemma, 124
arity map, 5
atomic
 model, 98
 strictly – models, 103
 theory, 99
automorphism, 9
axiomatization, 36

B

Baldwin-Lachlan theorem, 90
base, 88
Beth definability theorem, 130

C

categorical
 λ , 26
 ω , 26, 110
 countably, 26, 110
 totally, 26
chain, 10
 elementary, 25
characteristic
 of a field, 58
 of a structure, 8, 24
closed
 family of – subsets, 103

subset, 104
closure
 algebraic, 74
 definable, 74
 operator of, 88
codomain
 of a map, 8
compactness theorem
 general case, 40
 quantifier-free case, 38
complete
 $p(\bar{x})$ is complete over A , 37
 formula – over B , 98
consequence
 logical, 36
consistent
 $\varphi(\bar{x})$ is consistent in M , 19
 p is consistent in M , 36
 finitely, 36
constant
 function, 6
 relation, 6, 48
construction sequence, 99
carrier of a structure, 6

D

decide
 p decides φ , 37
definable
 closure, 74
 elements, 74
 set, 20
 subset, 20
 uniform family of – sets, 145
 uniformly, 145
dense
 family of closed subsets, 103
 linear order, 11
diagram
 – lemma, 31
 elementary, 31
 quantifier-free, 31
dimension, 88

domain
 of a map, 8
 of a structure, 6
domain of definition, 8

E

elementary
 1, 52
 amalgamation lemma, 124
 chain, 25
 diagram, 31
 embedding, 28
 equivalence, 24
 extension, 30
 homogeneous, 65
 map, 28
 substructure, 24
 superstructure, 24
elimination
 of imaginaries, 141
 uniform – of imaginaries, 145
 weak – of imaginaries, 141
elimination of quantifiers, 47
embedding, 9, 28
 elementary, 28
 joint – property, 122
equivalence
 elementary, 24
 finite – relation, 134
 formula, 134
existential
 formula, 52
 quantifier, 15
extension
 elementary, 30

F

false in M , 18
field
 algebraically closed, 58
 characteristic of a, 58
 ordered, 60
fields

real closed, 60
 finitely consistent, 36
 first-order, 16
 fixes

- pointwise, 9

 formula, 15

- \forall , 52
- \exists , 52

 atomic, 15
 closed, 15
 complete, 98
 equivalence, 134
 existential, 52
 mapping of a, 29
 parameter-free, 15
 quantifier-free, 15
 solution of a, 20
 universal, 52
 full

- p is full, 38

 function

- symbol for, 5

G

- generated substructure, 8, 29
- generic
 - structure, 118
- graph, 11

H

- holds
 - $\varphi(\bar{x})$ holds in M , 19
- homogeneous
 - λ , 65
 - elementary, 65
 - structure, 119

I

- independent set, 88
- inductive
 - family of closed subsets, 103
- interalgebraic

elements, 73
 interdefinable

- elements, 74
- sets and elements, 141

 interpolant, 131
 interpretation

- of formulas, 19
- of symbols of the language, 6
- of terms, 17

 isolate

- A isolates \bar{c} , 98
- A isolates \bar{c} over B , 98
- A isolates $p(\bar{x})$, 81
- $\varphi(\bar{x})$ isolates \bar{c} over B , 98
- $\varphi(\bar{x})$ isolates $p(\bar{x})$, 81

 isolated

- \bar{c} is – over A , 98
- element, 98
- type, 81

 isomorphism, 9

- partial, 28

L

- Löwenheim-Skolem theorem
 - downward, 33
 - upward, 40
- language
 - first-order, 5
 - many-sorted, 147
 - of fields, 10
 - of graphs, 11
 - of groups, 11
 - of rings, 10
 - relational, 8
- limit of a chain, 10
- link of a graph, 11
- localization, 9, 89
- logical
 - connectives, 15
 - consequence, 36

M

- map, 8

back-and-forth extendible, 65
 elementary, 28
 extendible, 65
 injective, 8
 inverse, 8
 surjective, 8
 total, 8
 minimal

- model (definability), 87
- model (inclusion), 102
- strongly – formula, 86
- strongly – set, 86
- strongly – theory, 85

 model, 6, 61, 118, 122

- M models T , 36
- M models φ , 18
- atomic, 98
- prime, 97
- strictly-atomic, 103
- strictly-prime, 99

 model-completeness, 52
 modulo

- modulo S , 47

 Morley theorem, 90
 morphism, 118

O

- omitting
 - a definable set, 79
 - a small set, 73
 - a type-definable set, 81
- orbit
 - of a set, 129
 - of a tuple, 68
- order
 - dense linear, 11

P

- parameter, 9, 14
 - canonical, 141
- prime
 - model, 97
 - strictly – model, 99

substructure, 8
 prime substructure, 8

Q

- quantifier
 - elimination, 47
 - existential, 15
 - universal, 19
- quantifier-free
 - amalgamation lemma, 124
 - diagram, 31

R

- range
 - of a map, 8
 - of a tuple, 7
- realization, 20, 36
- realize
 - M realizes p , 36
 - \bar{a} realizes p in M , 36
 - \bar{a} realizes $p(\bar{x})$ in M , 20
- relation
 - finite equivalence, 134
 - symbol for, 5

S

- saturated, 41
 - λ , 63
 - structure, 41
 - weakly, 70
- sentence, 15
- signature
 - first-order, 5
 - of fields, 10
 - of graphs, 11
 - of groups, 11
 - of rings, 10
 - relational, 8
- small, 118
 - cardinality, 61
 - theory, 64
- solution of a formula, 20

strictly
 atomic, 103
 prime, 99

structure
 finite, 26
 first-order, 6
 generic, 118
 homogeneous, 119
 many-sorted, 147
 universal, 119
 universal-homogeneous, 119

substitution
 of terms in formulas, 16
 of terms in terms, 14

substructure, 7
 elementary, 24
 generated, 8, 29
 prime, 8

superstructure
 elementary, 24

support
 of a map, 8
 of a structure, 6

T
 Tarski-Vaught test, 32

term, 14
 atomic, 14
 closed, 14
 parameter-free, 14

theory, 23
 atomic, 99
 complete, 23
 consistent, 23
 of M over A , 23
 quantifier-free, 23

true in M , 18

truth value, 18

tuple
 finite, 7
 infinite, 7
 range of, 7

type, 20, 36

A isolates $p(\bar{x})$, 81
 $\varphi(\bar{x})$ isolates $p(\bar{x})$, 81
 -definable set, 20
 isolated, 81
 of \bar{a} over A , 62
 transcendental, 87

U

universal
 domain, 61
 formula, 52
 quantifier, 19
 structure, 119

universal-homogeneous
 structure, 119

universe, 61

V

variable
 bound, 15
 free, 13

vector space, 11

vertex of a graph, 11

W

weakly saturated, 70

witness
 algebraic, 78
 of a formula, 20

Z

Zilber theorem, 113