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Uncertainty principle, positivity and L^p-boundedness for generalized spectrograms

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Abstract

In this paper we are concerned with the properties of positivity, uncertainty principle and continuity in L^p spaces of a generalized spectrogram. In particular we study the connections of a generalized spectrogram, as a subclass of the Cohen class, with the Rihaczek and the Wigner representations. We also consider the behavior of the generalized spectrogram with respect to the positivity and the L^p boundedness of the corresponding localization operators.

Keywords: Quadratic representations; Short-time Fourier transform; Spectrogram; Rihaczek representation; Wigner representation

1. Introduction

A time–frequency representation is a quadratic form which associates a signal f on $\mathbb{R}^d$ with a function (or distribution) Qf on the time–frequency plane $\mathbb{R}^d_x \times \mathbb{R}^d_\omega$. $Qf(x,\omega)$ represents the distribution of the energy of the signal with respect to the time variable x and the frequency variable ω and indicates therefore which frequencies ω are present in the signal f around the time x. In this context we shall use interchangeably the term "representation" or "form". It is generally required that $Q(f)$ satisfies some other conditions, namely:

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- *Positivity*: $Q(f)(x,\omega) \geqslant 0$ for all x, ω;
- *No spreading effect*: $\operatorname{supp} f \subseteq I$ for an interval $I \subseteq \mathbb{R}^d$ implies $\Pi_x \operatorname{supp} Q(f) \subseteq I$ ($\Pi_x =$ orthogonal projection $\mathbb{R}^d_x \times \mathbb{R}^d_\omega \to \mathbb{R}^d_x$) and, analogously, $\operatorname{supp} \hat{f} \subseteq J$ implies $\Pi_\omega \operatorname{supp} Q(f) \subseteq J$;
- *Marginal distributions condition*: $\int_{\mathbb{R}^d} Q(f)(x,\omega)\,dx = |\hat{f}(\omega)|^2$ and $\int_{\mathbb{R}^d} Q(f)(x,\omega)\,d\omega = |f(x)|^2$.

A description of the motivations and the meaning of these requirements can be found e.g. in [6]. As a fact related to the uncertainty principle, it turns out however that these conditions are incompatible and they can therefore be satisfied only with a certain degree of approximation. Many different representations have been defined in the literature in the attempt to approach as near as possible an ideal representation (see [7,10,11,14,15]).

Three of the most used time–frequency representations are the *spectrogram*, the *Rihaczek* and the *Wigner* representation. We recall their definitions and their main properties in Section 2.

On the other side time–frequency analysis is in many ways connected with the theory of pseudo-differential operators. For example, it is well known that the Wigner representation yields the class of Weyl operators via formula (2.6), whereas localization operators in (2.8), can be seen as filters for signals (see [8]).

In this paper we consider a quite natural "two-window" generalization of the spectrogram, which we call *generalized spectrogram* (Definition 2.1). Actually this representation already appeared implicitly in a number of works (see [1,3]), here however we explicitly study its properties and point out its basic role in the comprehension of the connections between time–frequency representations and operators.

More precisely in Section 2 we prove that the Rihaczek representation can be obtained as a generalized spectrogram with suitable distributional windows. We show then that, in an analogous way as the (cross) Wigner representation is connected with the class of Weyl operators, the (cross) generalized spectrogram yields the class of localization operators, whereas, as a limit case of localization operators, classical pseudo-differential operators are obtained from the (cross) Rihaczek representation.

In Section 3 we establish a convolution formula expressing the generalized spectrogram in terms of the Wigner representation and show therefore that the generalized spectrogram, as well as the Rihaczek representation, are included in the Cohen class [7].

On the other hand we also prove that generalized spectrograms do not cover all the Cohen class by showing explicitly that for example the Wigner representation does not belong to the generalized spectrogram class.

We turn then our attention to the corresponding operators and, as another consequence of the convolution formula, we obtain that positive symbols a yield positive localization operators $L^a_{\phi,\psi}$ if and only if $\phi = \psi$.

In Section 4, we extend estimates of Lieb [16] to the generalized spectrogram and we prove in this context a natural extension of Lieb's uncertainty principle.

The final Section 5 is devoted to complete the study of localization operators with $L^p(\mathbb{R}^{2d})$ symbols on $L^q(\mathbb{R}^d)$ spaces, which was started in [3]. We show that the boundedness results contained there can be proved more directly using the properties of the generalized spectrogram. Further we show that the conditions in [3] are actually necessary and sufficient for boundedness, i.e. nonboundedness holds in the remaining cases. This yields a complete picture of the $L^p(\mathbb{R}^d)$ boundedness properties of localization operators with symbols in $L^q(\mathbb{R}^{2d})$.

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2. Forms and operators, the generalized spectrogram

We shall consider here distributions as antilinear functionals, so that the L^2 product (u, v) extends to the action of a distribution u on a test function v.

We revise at first some facts about *spectrograms*, *Rihaczek* and *Wigner* representations.

The definition of the spectrogram relies on the *Gabor transform* (also *short-time Fourier transform* or, for short, *STFT*) $V_\phi f(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i t\omega}\overline{\phi(t-x)}f(t)\,dt$ of a signal f and a "window" ϕ, whose action is a localization of the signal f in time by multiplication with translations of $\phi(t)$, before taking its Fourier transform. The conjugation on the window appears just for mathematical convenience in such a way that $V_\phi f(x,\omega) = (f, \phi_{x,\omega})_{L^2}$, where $\phi_{x,\omega}(t) = e^{2\pi i\omega t}\phi(t-x)$.

The *spectrogram* is then defined as

$$\mathrm{Sp}_\phi(f)(x,\omega) = \left|V_\phi f(x,\omega)\right|^2. \tag{2.1}$$

It is of course a positive distribution but it does not satisfy the marginals and has a spreading effect depending on the support of the window ϕ (see [6,12]).

The *Rihaczek* quadratic representation is essentially defined as the product of the signal $f(x)$ with its Fourier transform $\hat{f}(\omega)$, more precisely it is the distribution

$$R(f)(x,\omega) = e^{-2\pi i x\omega} f(x)\overline{\hat{f}(\omega)}. \tag{2.2}$$

Despite its elementary definition it has reasonable physical motivations and was widely used in the time–frequency analysis of signals (see [6,13]). As one can immediately verify, it satisfies the marginals and has no spreading effect, however it is evidently not positive.

The third form we want to consider is the *Wigner* representation (see [20])

$$\mathrm{Wig}(f)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i x\omega} f(x+t/2)\overline{f(x-t/2)}\,dt, \tag{2.3}$$

defined by Wigner in the context of quantum theory. As the Rihaczek representation, it is not positive but it has no spreading effect and it satisfies the marginals (see [6,12,14]).

Of course, by polarization, all three quadratic forms are associated with corresponding cross (i.e. sesquilinear) forms:

- $\mathrm{Sp}_\phi(f,g)(x,\omega) = V_\phi f\,\overline{V_\phi g}(x,\omega)$ (*cross* spectrogram) (ϕ fixed window);
- $R(f,g)(x,\omega) = e^{-2\pi i x\omega} f(x)\overline{\hat{g}(\omega)}$ (*cross Rihaczek* distribution);
- $\mathrm{Wig}(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i x\omega} f(x+t/2)\overline{g(x-t/2)}\,dt$ (*cross Wigner* distribution).

They all define continuous maps $S(\mathbb{R}^d) \times S(\mathbb{R}^d) \to S(\mathbb{R}^{2d})$ which extend continuously to $S'(\mathbb{R}^d) \times S'(\mathbb{R}^d) \to S'(\mathbb{R}^{2d})$.

We define next the principal types of pseudo-differential operators and investigate their basic relations to the above-mentioned sesquilinear representations.

A (classical) pseudo-differential operator A^a with symbol a, for simplicity for the moment $a \in S(\mathbb{R}^{2d})$, is the map on $S(\mathbb{R}^d)$ defined as

$$A^a f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\omega} a(x,\omega)\hat{f}(\omega)\,d\omega. \tag{2.4}$$

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Writing explicitly the Fourier transform this expression becomes $A^a f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\omega} \times a(x,\omega) f(y)\,dy\,d\omega$ and it appears therefore natural to allow symbols to depend more generally also on y, i.e. one is led to consider operators of the form $A^\sigma f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\omega} \sigma(x,y,\omega) \times f(y)\,dy\,d\omega$. In the case where $\sigma(x,y,\omega) = a((x+y)/2,\omega)$ we have the important case of *Weyl operators*

$$W^a f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\omega} a\left(\frac{x+y}{2}, \omega\right) f(y)\,dy\,d\omega \tag{2.5}$$

and a is called the *Weyl symbol* of W^a.

The first important connection between pseudo-differential operator theory and quadratic time–frequency representation is given by the well-known formula

$$\big(W^a f, g\big) = \big(a, \mathrm{Wig}(g,f)\big). \tag{2.6}$$

The integrals in (2.4) and (2.5), and the L^2 product in (2.6), as well as in the following, should be intended in a week sense in the more general case $a \in \mathcal{S}'(\mathbb{R}^{2d})$.

In the particular case where Weyl operators have symbols b of the form $b = a * \mathrm{Wig}(\psi,\phi)$, they are called *localization operators* $L^a_{\phi,\psi}$ with *symbol* a, *analysis window* ϕ and *reconstruction window* ψ. In other words

$$L^a_{\phi,\psi} = W^{a*\mathrm{Wig}(\psi,\phi)}. \tag{2.7}$$

We recall that the Wigner transform of gaussians is still a function of gaussian type. More precisely, in the particular case $\phi(x) = \psi(x) = e^{-\pi x^2} =: g(x)$, we have $L^a_{g,g} = W^{a*G}$ with $G(x,\omega) = 2^{d/2} g(\sqrt{2}x) g(\sqrt{2}\omega)$. In this case localization operators are also known as *anti-Wick operators* and were used in pseudo-differential operators theory as approximations of general Weyl operators (see [17]).

In time–frequency analysis however localization operators originated independently as filters for signals based on the Gabor transform and this constitutes another very basic connection between pseudo-differential operators theory and signal analysis. Namely a straightforward computation shows that the operator $L^a_{\phi,\psi}$ in (2.7) has the form

$$L^a_{\phi,\psi} f(s) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_\phi f(x,\omega) \psi_{x,\omega}(s)\,dx\,d\omega \tag{2.8}$$

where $\psi_{x,\omega}(s)(t) = e^{2\pi i t\omega}\psi(t-x)$. The three step analysis–processing–reconstruction of the signal f are represented in (2.8) as Gabor transform, multiplication with $a(x,\omega)$, and integration against $\psi_{x,\omega}(s)$, respectively. For references about this subject, see e.g. [2,8,18,19,21,22].

From (2.8) it is clear that

$$\big(L^a_{\phi,\psi} f, g\big) = \big(a, V_\psi g \overline{V_\phi f}\big). \tag{2.9}$$

If we compare (2.9) with (2.6) we see that it has the same structure with the Wigner transform replaced by the form $V_\psi g \overline{V_\phi f}$. It appears then natural to introduce the following definition.

Definition 2.1. The *generalized spectrogram*, depending on the two windows ϕ and ψ, is defined as the sesquilinear form

$$q_{\psi,\phi}(g,f)(x,\omega) = V_\psi g(x,\omega) \overline{V_\phi f}(x,\omega). \tag{2.10}$$

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For what concerns its functional setting we remark at first that the generalized spectrogram is a well-defined function in $\mathcal{S}(\mathbb{R}^{2d})$ if f, g, ϕ, ψ are in $\mathcal{S}(\mathbb{R}^d)$ and it belongs to $L^1(\mathbb{R}^{2d})$ if $f, g, \phi, \psi \in L^2(\mathbb{R}^d)$. As the Gabor transform extends to tempered distributions, we are however allowed, at least in some cases, to consider in (2.10) distributional windows.

As a first result we show then that, not only the (classical) spectrogram, but also the Rihaczek representation can be obtained as a particular case of generalized spectrogram.

Proposition 2.1.

(i) $q_{\phi,\phi}(f, g) = \mathrm{Sp}_\phi(f, g)$, *for* $f, g \in S(\mathbb{R}^d)$, $\phi \in \mathcal{S}'(\mathbb{R}^d)$;
(ii) $q_{\delta,1}(f, g) = R(f, g)$, *for* $f, g \in S(\mathbb{R}^d)$.

Proof. (i) is trivial. (ii) Consider the limit cases $\phi = \delta$, the point measure $(\delta, \varphi) = \overline{\varphi(0)}$ and $\psi = 1$. For $\chi \in \mathcal{S}(\mathbb{R}^{2d})$ we have

$$\begin{aligned}(V_\delta f, \chi) &= \big(\mathcal{F}_2\tau_a(f \otimes \bar{\delta}), \chi\big) = \big(f \otimes \bar{\delta}, \tau_a^{-1}\mathcal{F}_2^{-1}\chi\big) = \big(f, \overline{(\bar{\delta}, \tau_a^{-1}\mathcal{F}_2^{-1}\chi)}\big) \\ &= \left(f, \int_{\mathbb{R}^d} e^{2\pi i x\omega}\chi(x - t, \omega)\, d\omega\bigg|_{t=0}\right) = \left(f, \int_{\mathbb{R}^d} e^{2\pi i x\omega}\chi(x, \omega)\, d\omega\right) \\ &= \int_{\mathbb{R}^{2d}} e^{-2\pi i x\omega} f(x)\overline{\chi(x, \omega)}\, dx\, d\omega = \big(e^{-2\pi i x\omega} f(x), \chi(x, \omega)\big).\end{aligned}$$

Therefore

$$V_\delta f(x, \omega) = e^{-2\pi i x\omega} f(x). \tag{2.11}$$

Now by relation (2.11) we obtain

$$V_1 g(x, \omega) = e^{-2\pi i x\omega} V_{\hat{1}}\hat{g}(\omega, -x) = e^{-2\pi i x\omega} V_\delta \hat{g}(\omega, -x) = \hat{g}(\omega).$$

We conclude that $V_\delta f\, \overline{V_1 g}(x, \omega) = e^{-2\pi i x\omega} f(x)\overline{\hat{g}(\omega)}$. □

Classical spectrograms and Rihaczek representations constitute actually "extremes" points of the generalized spectrogram. From the point of view of the applications let us start by considering a (classical) spectrogram $|V_\phi f|^2 = V_\phi f\,\overline{V_\phi f}$. We can dilate the window of one term and contract the other in such a way that the dilated window tends in $\mathcal{S}'(\mathbb{R}^d)$ to the constant 1 and the other window to the point distribution δ, positivity is then lost but we have an improvement in the limitation of the spreading effect and the marginal conditions toward the extreme case of the Rihaczek where no spreading effect is present and the marginals are satisfied.

The generalized spectrogram $q_{\phi,\psi}(f, g) = V_\phi f\,\overline{V_\psi g}$ represents therefore a link between the spectrogram $\mathrm{Sp}_\phi(f, g)$ and the Rihaczek distribution $R(f, g)$: an explicit "path" with gaussian windows is for example $q_{\phi_\lambda,\hat{\phi}_\lambda}(f, g)$ where $\phi_\lambda(x) = \lambda^{d/2}e^{-\pi\lambda x^2}$, $\lambda \in [1, \infty]$, with the convention $\phi_\infty = \delta$, $\psi_\infty = \hat{\phi}_\infty = 1$.

In Section 3 after we have deduced a convolution formula for the generalized spectrogram we shall be able to give a negative answer to the natural question on whether the Wigner representation could possibly constitute also a particular case of the generalized spectrogram.

As next step we consider now (2.9) in the case $\phi = \delta$, $\psi = 1$, i.e., according to the previous proposition, in the case where the generalized spectrogram coincides with the Rihaczek repre-

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sentation. We see with the next proposition that in this way we recover the class of (classical) pseudo-differential operators.

Proposition 2.2. *The operator class associated with the Rihaczek representation by* (2.9) *is the class of pseudo-differential operators* $A^a f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\omega} a(x,\omega)\hat{f}(\omega)\,d\omega$.

Proof. Let $a \in S'(\mathbb{R}^{2d})$, $f, g \in S(\mathbb{R}^d)$. For the operator T_R^a associated with $R(g, f)$ by (2.9) we have then

$$\begin{aligned}\left(T_R^a f, g\right) = \left(a, R(g, f)\right) &= \left(a, e^{-2\pi i x\omega} g(x)\overline{\hat{f}(\omega)}\right)\\ &= \int_{\mathbb{R}^{2d}} e^{2\pi i x\omega} a(x,\omega)\overline{g(x)}\hat{f}(\omega)\,dx\,d\omega\\ &= \left(\int_{\mathbb{R}^d} e^{2\pi i x\omega} a(x,\omega)\hat{f}(\omega)\,d\omega, g\right). \qquad \square\end{aligned}$$

As a consequence of this proposition, classical pseudo-differential operators can be regarded as a particular case of localization operators, just as the Rihaczek representation is a particular case of the generalized spectrogram, completing therefore our symmetrical picture of forms and operators.

3. A convolution formula, generalized spectrogram as subclass of Cohen class and positivity of localization operators

Let us consider again formulas (2.6) and (2.9). As shown in the previous section they define Weyl and localization operators by means of the corresponding Wigner and generalized spectrogram representations, respectively. On the other hand localization operators are expressed as Weyl operators through the well-known relation (2.7). This means that for every $\phi, \psi, f, g \in S(\mathbb{R}^d)$, defining the reflection $\tilde{\psi}(x) = \psi(-x)$, we have

$$\begin{aligned}(a, V_\psi g\overline{V_\phi f}) &= \left(L^a_{\phi,\psi} f, g\right) = \left(W^{a*\mathrm{Wig}(\psi,\phi)} f, g\right)\\ &= \left(a * \mathrm{Wig}(\psi,\phi), \mathrm{Wig}(g, f)\right) = \left(a, \mathrm{Wig}(\tilde{\psi},\tilde{\phi}) * \mathrm{Wig}(g, f)\right).\end{aligned}$$

As this holds for every $a \in S'(\mathbb{R}^{2d})$, we obtain the following convolution formula

$$V_\psi g\overline{V_\phi f} = \mathrm{Wig}(\tilde{\psi},\tilde{\phi}) * \mathrm{Wig}(g, f) \tag{3.1}$$

by which the generalized spectrogram is expressed as convolution of Wigner transforms.

We recall now the following definition.

Definition 3.1. The *Cohen class* of time–frequency representations is defined as the class of sesquilinear forms of the type $\sigma * \mathrm{Wig}(g, f)$ with $\sigma \in S'(\mathbb{R}^{2d})$.

An immediate consequence of the previous convolution formula is therefore that the generalized spectrogram representation, where more generally we can suppose $\phi, \psi \in S'(\mathbb{R}^{2d})$ in any case where it makes sense, is a subclass of the Cohen class.

Another consequence of (3.1) is that Rihaczek form, which we expressed as particular generalized spectrogram, can be written as

$$R(g,f)=\mathrm{Wig}(1,\delta)*\mathrm{Wig}(g,f) \tag{3.2}$$

or more explicitly

$$R(g,f)=e^{-4\pi i x\omega}*\mathrm{Wig}(g,f)(x,\omega). \tag{3.3}$$

It is now interesting to consider the question if also the Wigner representation, which is expressed in the Cohen class with the trivial choice $\sigma=\delta$, could possibly be expressed as a generalized spectrogram. This would be the case if $\mathrm{Wig}(\psi,\phi)=\delta$ for some windows ψ,ϕ. With regard to the general concept, underlying the uncertainty principle, that the support of a time–frequency representation cannot be too "small," one can at once argue that the answer is negative. This is precisely proved by the following proposition.

Proposition 3.1. *Let* $f,g\in S'(\mathbb{R}^d)$, *then* $\mathrm{Wig}(f,g)\neq\delta\in S'(\mathbb{R}^{2d})$.

Proof. Suppose on the contrary that there exist $f,g\in S'(\mathbb{R}^d)$ such that $\mathrm{Wig}(f,g)=\delta$. Of course $f,g\neq 0$ otherwise $\mathrm{Wig}(f,g)=0$. Let us factorize the Wigner form as $\mathrm{Wig}(f,g)=\mathcal{F}_2(Ts(f\otimes\bar g))$ where $Ts:\Psi(x,t)\in S'(\mathbb{R}^{2d})\to(Ts\Psi)(x,t)=\Psi(x+t/2,x-t/2)\in S'(\mathbb{R}^{2d})$, $\mathcal{F}_2$ is the partial Fourier transform with respect to the second d variables, and we recall that as antilinear functional the conjugate of a distribution $g\in\mathcal{S}'(\mathbb{R}^d)$ is defined by $(\bar g,\phi)=\overline{(g,\bar\phi)}$, $\phi\in\mathcal{S}(\mathbb{R}^d)$. Using the fact that $\delta=\delta_1\otimes\delta_2$ with $\delta_j\in S'(\mathbb{R}^d)$ $(j=1,2)$, we have

$$\delta_1\otimes\delta_2=\mathcal{F}_2\big(Ts(f\otimes\bar g)\big).$$

Taking the partial inverse Fourier transform

$$\delta_1\otimes 1=Ts(f\otimes\bar g)$$

and therefore

$$(Ts)^{-1}(\delta_1\otimes 1)=f\otimes\bar g.$$

This means that for every $\phi_1,\phi_2\in S(\mathbb{R}^d)$ we have

$$\begin{aligned}(f,\phi_1)\overline{(g,\overline{\phi_2})}&=(f\otimes\bar g,\phi_1\otimes\phi_2)=\big((Ts)^{-1}(\delta_1\otimes 1),\phi_1\otimes\phi_2\big)=\big(\delta_1\otimes 1,Ts(\phi_1\otimes\phi_2)\big)\\&=\overline{\int_{\mathbb{R}^d}\phi_1(t/2)\overline{\phi_2(-t/2)}\,dt}=2^d\int_{\mathbb{R}^d}\overline{\phi_1(s)}\tilde\phi_2(s)\,ds.\end{aligned}$$

As $g\neq 0$ there exists $\phi_2\in S(\mathbb{R}^d)$ such that $(g,\overline{\phi_2})\neq 0$ so we can write

$$(f,\phi_1)=\frac{2^d}{\overline{(g,\overline{\phi_2})}}\int_{\mathbb{R}^d}\overline{\phi_1(s)}\tilde\phi_2(s)\,ds \tag{3.4}$$

for $\phi_1\in S(\mathbb{R}^d)$. This means that the distribution f coincides with the function $f(s)=\frac{2^d}{\overline{(g,\overline{\phi_2})}}\tilde\phi_2(s)$ for every ϕ_2 for which $(g,\overline{\phi_2})\neq 0$. This is absurd, namely, take for instance $i\phi_2$ instead of ϕ_2, then $f(s)=-\frac{2^d}{\overline{(g,\overline{\phi_2})}}\tilde\phi_2(s)$, which would mean $f=0$. □

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We remark that the previous proposition also shows that the generalized spectrogram, even allowing distributional windows, do not cover all the Cohen class.

We can summarize with the following scheme the frame we have constructed:

Sesquilinear form	$\sigma * \mathrm{Wig}(g, f)$ (Cohen cl.)	Operator
$\mathrm{Wig}(g, f)$ (Wigner)	$\delta * \mathrm{Wig}(g, f)$	W^a (Weyl)
$q_{\psi,\phi}(g, f) = V_\psi g \overline{V_\phi f}$ (gen. spectr.)	$\mathrm{Wig}(\tilde{\psi}, \tilde{\phi}) * \mathrm{Wig}(g, f)$	$L^a_{\phi,\psi}$ (localization)
$R(g, f)$ $(= q_{\delta,1}(g, f))$ (Rihaczek)	$e^{-4\pi x\omega} * \mathrm{Wig}(g, f)$	A^a (classical ΨDO)

(3.5)

We prove next a result on positivity for localization operators, after giving two preliminary results.

Proposition 3.2. *Let $f, g \in \mathcal{S}'$ be such that* $\mathrm{Wig}(f, g) * \mathrm{Wig}(u, u) \geqslant 0$, $u \in \mathcal{S}'$. *Then* $\mathrm{Wig}(f, g)(x, \omega)$ *is real.*

Proof. We have that $(\mathrm{Re}\,\mathrm{Wig}(f, g) + i\,\mathrm{Im}\,\mathrm{Wig}(f, g)) * \mathrm{Wig}(u, u) \geqslant 0$. Since $\mathrm{Wig}(u, u)$ is real, we write $(\mathrm{Re}\,\mathrm{Wig}(f, g) * \mathrm{Wig}(u, u) + i\,\mathrm{Im}\,\mathrm{Wig}(f, g) * \mathrm{Wig}(u, u)) \geqslant 0$. Then $\mathrm{Im}\,\mathrm{Wig}(f, g) * \mathrm{Wig}(u, u) = 0$, that means $\mathcal{F}(\mathrm{Im}\,\mathrm{Wig}(f, g)) \cdot \mathcal{F}(\mathrm{Wig}(u, u)) = 0$, for every $u \in \mathcal{S}'$. Let now u be such that $\mathcal{F}\,\mathrm{Wig}(u, u) \neq 0$ a.e. in $\mathbb{R}^2$ (for example for u gaussian function), then $\mathcal{F}(\mathrm{Im}\,\mathrm{Wig}(f, g)) = 0$ implies $\mathrm{Im}\,\mathrm{Wig}(f, g) = 0$, and $\mathrm{Wig}(f, g)(x, \omega)$ is real. □

Proposition 3.3. *Let $f, g \in \mathcal{S}'$. Then $\mathrm{Wig}(f, g)(x, \omega)$ is a real function if and only if $f = Cg$, with C real constant.*

Proof. Since $\mathrm{Wig}(f, g) = \overline{\mathrm{Wig}(g, f)}$ for every $f, g \in \mathcal{S}'$, when $f = Cg$ we easily get that $\mathrm{Wig}(f, g)$ is real. On the other side, if $\mathrm{Wig}(f, g)$ is real, $\mathrm{Wig}(f, g) = \mathrm{Wig}(g, f)$ holds. It means that $\mathcal{F}_2\tau_a(f \otimes \bar{g}) = \mathcal{F}_2\tau_a(g \otimes \bar{f})$, that is $(f \otimes \bar{g}) = (g \otimes \bar{f})$. Since by definition of tensorial product of two distributions we have $(h \otimes k)(\phi_1 \otimes \phi_2) = (h, \phi_1)(k, \phi_2)$, we get $f = Cg$. □

Theorem 3.1 *(Positivity). The Localization operator $L^a_{\phi,\psi}$ is positive, i.e. for all positive symbols a, $L^a_{\phi,\psi} \geqslant 0$ if and only if there exists a constant C such that $\phi = C\psi$.*

Proof. We recall that

$$\big(L^a_{\phi,\psi} f, f\big) = (a, V_\psi f \overline{V_\phi f}) = \big(a, \mathrm{Wig}(\tilde{\psi}, \tilde{\phi}) * \mathrm{Wig}(f, f)\big).$$

For $\phi = C\psi$ we obtain that $a \geqslant 0$ implies $L^a_{\phi,\psi} \geqslant 0$. In the other sense, if $\mathrm{Wig}(\tilde{\psi}, \tilde{\phi}) * \mathrm{Wig}(f, f)$ is positive, it suffices to apply Propositions 3.2 and 3.3. □

4. Generalized spectrogram and uncertainty principle

The Gabor transform extends to a map from $S'(\mathbb{R}^d) \times S'(\mathbb{R}^d)$ to $S'(\mathbb{R}^{2d})$ but the generalized spectrogram $q_{\psi,\phi}(g, f) = V_\psi g \overline{V_\phi f}$ is in general not defined if $V_\phi f$ and $V_\psi g$ are in $S'(\mathbb{R}^{2d})$. In order to specify its action in the case of possibly distributional windows we need then to introduce some restrictions.

We shall give three settings in which the generalized spectrogram can be defined: distributions, C^∞ functions and L^p functions.

For an interpretation in the frame of tempered distribution we proceed as follows.

Definition 4.1. Let $\mathcal{B}^\infty(\mathbb{R}^d)$ be the space of smooth bounded functions together with all its derivatives, i.e. the space of C^∞ functions h such that for every multiindex $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ there exists a constant M_α with $|\partial_x^\alpha h(x)| \leqslant M_\alpha$ on $\mathbb{R}^d$.

We observe that $q_{\psi,\phi}(g, f) = V_\psi g \overline{V_\phi f}$ makes sense whenever $V_\psi g \in S'(\mathbb{R}^{2d})$ and $V_\phi f \in \mathcal{B}^\infty(\mathbb{R}^d)$ or vice versa.

Lemma 4.1. *If $\phi \in \mathcal{B}^\infty(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$, then $V_\phi f \in \mathcal{B}^\infty(\mathbb{R}^{2d})$.*

Proof. By the differentiation under the integral, by the boundedness of ϕ together of all the derivatives $\partial_x^\alpha \phi$, and since $f_\beta = (-2\pi i)^{|\beta|} t^{|\beta|} f(t) \in \mathcal{S}(\mathbb{R}^d)$, $|\beta| = \beta_1 + \cdots + \beta_d$, we have that for any couple of multi-indices α, β, there exist a constant M_α such that

$$\left|\partial_x^\alpha \partial_\omega^\beta V_\phi f\right| \leqslant \int_{\mathbb{R}^d} \left|f_\beta(t)\right| \left|\partial_x^\alpha \phi(t-x)\right| dt \leqslant M_\alpha \|f_\beta\|_{L_1},$$

which gives continuity and boundedness of $V_\phi f$ on $\mathbb{R}^{2d}$ together with all its derivatives. □

(Of course the role of f and ϕ in Lemma 4.1 can be exchanged.)

As a consequence of Lemma 4.1 we have the following proposition.

Proposition 4.1. *Let $g, \psi \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$, $\phi \in \mathcal{B}^\infty(\mathbb{R}^d)$ (or vice versa $f \in \mathcal{B}^\infty(\mathbb{R}^d)$, $\phi \in \mathcal{S}(\mathbb{R}^d)$). Then $q_{\psi,\phi}(g, f)$ is a well defined tempered distribution in $\mathcal{S}'(\mathbb{R}^{2d})$.*

Next in order to obtain regularity for the generalized spectrogram $q_{\psi,\phi}$ we remark that $V_\psi g(x, \omega) = \int_{\mathbb{R}^d} g(s + x) e^{-2\pi i(s+x)\omega} \overline{\psi(s)}\, ds$. If $g \in \mathcal{S}(\mathbb{R}^d)$ then $g(s + x) e^{-2\pi i(s+x)\omega} \in C^\infty(\mathbb{R}^d{}_s \times \mathbb{R}^d{}_x \times \mathbb{R}^d{}_\omega)$, and for fixed x, ω, we have $g(\cdot + x) e^{-2\pi i(.+x)\omega} \in \mathcal{S}(\mathbb{R}^d)$. It follows that $V_\psi g \in C^\infty(\mathbb{R}^{2d})$ whenever $g \in \mathcal{S}(\mathbb{R}^d)$ and $\psi \in \mathcal{S}'(\mathbb{R}^d)$. This proves the following proposition.

Proposition 4.2. *Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $\phi, \psi \in \mathcal{S}'(\mathbb{R}^d)$ (or vice versa $f, g \in \mathcal{S}'(\mathbb{R}^d)$, and $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$), then $q_{\psi,\phi}(g, f) \in C^\infty(\mathbb{R}^{2d})$.*

We consider finally the L^p-space setting. We shall need the following definition and the boundedness result in Lemma 4.2.

Definition 4.2. We say that a function $F(x, t)$, $(x, t) \in \mathbb{R}^{2d}$, belongs to the mixed $L_{x,t}^{p,q}(\mathbb{R}^d \times \mathbb{R}^d)$ space if $F(x, t)$ is an L^q function with respect to the variable t and $(\int |F(x, t)|^q\, dt)^{\frac{1}{q}}$ belongs to the L^p space with respect to the x variable. The quantity

$$\left\|F(x, t)\right\|_{p,q} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left|F(x, t)\right|^q dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty$$

represents a norm in $L_{x,t}^{p,q}(\mathbb{R}^d \times \mathbb{R}^d)$ for $F(x, t)$.

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Lemma 4.2. *If $f \in L^p(\mathbb{R}^d)$ and $\phi \in L^{p'}(\mathbb{R}^d)$, $\frac{1}{p} + \frac{1}{p'} = 1$, then $F(x,t) = f(t)\phi(t-x) \in L^{q,q'}_{x,t}(\mathbb{R}^{2d})$, with $\frac{1}{q} + \frac{1}{q'} = 1$ and $1 \leqslant q' \leqslant \min\{p, p'\}$. Also*

$$\|F(x,t)\|_{q,q'} = \left(\big\||f|^{q'} * |\phi|^{q'}\big\|_{\frac{q}{q'}}\right)^{\frac{1}{q'}}.$$

Proof. Let us first consider the case $1 < q' \leqslant \min\{p, p'\}$, the means $p \neq 1$ or $p \neq \infty$. Since $f \in L^p$ and $q' \leqslant p$, the function $|f|^{q'} \in L^{\frac{p}{q'}}(\mathbb{R}^d)$. The same arguments on the function ϕ show that $|\phi|^{q'} \in L^{\frac{p'}{q'}}(\mathbb{R}^d)$. From Young's inequality with the triple $(\frac{p}{q'}, \frac{p'}{q'}, \frac{q}{q'})$, $\frac{q'}{p} + \frac{q'}{p'} = 1 + \frac{q'}{q}$ we have that $|f|^{q'} * |\phi|^{q'} \in L^{\frac{q}{q'}}(\mathbb{R}^d)$. Now

$$\begin{aligned}
\infty > \left(\big\||f|^{q'} * |\phi|^{q'}\big\|_{\frac{q}{q'}}\right)^{\frac{1}{q'}} &= \left(\int_{\mathbb{R}^d} \left(|f|^{q'} * |\phi|^{q'}\right)^{\frac{q}{q'}}(x)\,dx\right)^{\frac{q'}{q}\frac{1}{q'}} \\
&= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \big|f(t)\phi(t-x)\big|^{q'}\,dt\right)^{\frac{q}{q'}}(x)\,dx\right)^{\frac{1}{q}} \\
&= \|F(x,t)\|_{q,q'}. \qquad \square
\end{aligned}$$

The same argument can be replaced in the case $p = 1$ or $p = \infty$ adopting the suitable definitions of norm. We also observe that the assumption $1 \leqslant q' \leqslant \min\{p, p'\}$ can be replaced with the analogous one $\max\{p, p'\} \leqslant q \leqslant \infty$.

Remark 4.1. Moreover, from Young inequality, we can obtain the optimal estimate

$$\big\||f|^{q'} * |\phi|^{q'}\big\|_{\frac{q}{q'}} \leqslant \left(C_{\frac{p}{q'}} C_{\frac{p'}{q'}} C_{(\frac{q}{q'})'}\right)^d \big\||f|^{q'}\big\|_{\frac{p}{q'}} \big\||\phi|^{q'}\big\|_{\frac{p'}{q'}}.$$

Remark 4.2. Taking the Fourier transform $\mathcal{F}_2 F(x,t)$, of the function $F(x,t)$, with respect to the second-variable, from Hausdorff–Young inequality follows that $\mathcal{F}_2 F(x,t) \in L^q(\mathbb{R}^d)$, and for almost all $x \in \mathbb{R}^d$,

$$\big\|\mathcal{F}_2 F(x,t)(x,\cdot)\big\|_q(x) \leqslant C_{q'} \big\|F(x,\cdot)\big\|_{q'}(x).$$

Theorem 4.1. *Let us fix p_j, p'_j, q_j, $j = 1, 2$, with $\frac{1}{p_j} + \frac{1}{p'_j} = 1$ and $q_j \geqslant \max\{p_j, p'_j\}$. If $f \in L^{p_1}$, $\phi \in L^{p'_1}$, $g \in L^{p_2}$, $\psi \in L^{p'_2}$ and $p = \frac{q_1 q_2}{q_1 + q_2}$, then*

(a) $$h \leqslant p < \infty, \quad \text{for } h = \frac{\max\{p_1, p'_1\}\max\{p_2, p'_2\}}{\max\{p_1, p'_1\} + \max\{p_2, p'_2\}};$$

(b) $$\iint_{\mathbb{R}^{2d}} \big|V_\psi g \overline{V_\phi f}(x,\omega)\big|^p\,dx\,dw \leqslant \left(\prod_{j=1}^{2} Q_j P_j\right)^{dp} \left(\|f\|_{p_1}\|g\|_{p_2}\|\phi\|_{p'_1}\|\psi\|_{p'_2}\right)^p,$$

where

$$Q_j = q_j^{-\frac{1}{q_j}} (q_j - 2)^{\frac{2-q_j}{2q_j}} \quad \text{and}$$

$$P_j = (p_j - 1)^{\frac{1-p_j}{2p_j}} p_j^{\frac{1}{q_j}} \left(q_j(p_j - 1) - p_j\right)^{\frac{q_j(p_j-1)-p_j}{2p_jq_j}} (q_j - p_j)^{\frac{q_j-p_j}{2p_jq_j}}.$$

Proof. First we observe that the function $p(q_1, q_2) = \frac{q_1q_2}{q_1+q_2}$, defined for $q_j \geqslant \max\{p_j, p_j'\}$, $j = 1, 2$, is unbounded, for instance if $q_1 = q_2$, and achieved its minimum for $(q_1, q_2) = (\max\{p_1, p_1'\}, \max\{p_2, p_2'\})$. This means that $p \geqslant \frac{\max\{p_1,p_1'\}\max\{p_2,p_2'\}}{\max\{p_1,p_1'\}+\max\{p_2,p_2'\}}$ holds, proving the statement (a).

Let now $F_1(x, t) = g(t)\bar{\psi}(t - x)$ and $F_2(x, t) = \bar{f}(t)\phi(t - x)$. Applying the Lemma 4.2 to the functions $F_j(x, t)$, $j = 1, 2$, we have that $F_j(x, t) \in L^{q_j,q_j'}_{x,t}$, $j = 1, 2$, $1 \leqslant q_j' \leqslant \min\{p_j, p_j'\} \leqslant 2$, where q_j, are the conjugate indexes of q_j'. We note that $V_\psi g(x, \omega) = \mathcal{F}_2F_1(x, \cdot)(x, \omega)$ and $\overline{V_\phi f}(x, \omega) = \mathcal{F}_2F_2(x, \cdot)(x, -\omega)$ for $x \in \mathbb{R}^d$.

The generalized Hölder inequality with the triple q_1, q_2, p, with $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p}$, that means $p = \frac{q_1q_2}{q_1+q_2}$, implies that for almost all $x \in \mathbb{R}^d$,

$$\begin{aligned}\int_{\mathbb{R}^d} \left|V_\psi g\overline{V_\phi f}(x, \omega)\right|^p dw &= \int_{\mathbb{R}^d} \left|\mathcal{F}_2F_1(x, t)(x, \omega) \cdot \mathcal{F}_2F_2(x, t)(x, -\omega)\right|^p dw \\ &\leqslant \left\|\mathcal{F}_2F_1(x, t)\right\|^p_{q_1} \left\|\mathcal{F}_2F_2(x, t)\right\|^p_{q_2}(x). \qquad (4.1)\end{aligned}$$

Hence, by (4.1), Remark 4.2 and Lemma 4.2, we have:

$$\int_{\mathbb{R}^d} \left|V_\psi g\overline{V_\phi f}(x, \omega)\right|^p dw \leqslant C^{dp}_{s_1}C^{dp}_{s_2}\left(|f|^{s_1} * |\tilde{\phi}|^{s_1}\right)^{\frac{p}{s_1}} \cdot \left(|\bar{g}|^{s_2} * |\tilde{\bar{\psi}}|^{s_2}\right)^{\frac{p}{s_2}}(x),$$

with the notations $s_j = q_j'$, $j = 1, 2$.

Let us also observe that $(|f|^{s_1} * |\tilde{\phi}|^{s_1})^{\frac{p}{s_1}} \in L^{\frac{q_1}{p}}$ and $(|\bar{g}|^{s_2} * |\tilde{\bar{\psi}}|^{s_2})^{\frac{p}{s_2}} \in L^{\frac{q_2}{p}}$ since $q_j \geqslant p = \frac{q_1q_2}{q_1+q_2}$. The generalized Hölder inequality with the triple $(\frac{q_1}{p}, \frac{q_2}{p}, 1)$, $\frac{p}{q_1} + \frac{p}{q_2} = p \cdot \frac{1}{p} = 1$ gives

$$\begin{aligned}\left\|\left(|f|^{s_1} * |\tilde{\phi}|^{s_1}\right)^{\frac{p}{s_1}} \cdot \left(|\bar{g}|^{s_2} * |\tilde{\bar{\psi}}|^{s_2}\right)^{\frac{p}{s_2}}\right\|_1 &\leqslant \left\|\left(|f|^{s_1} * |\tilde{\phi}|^{s_1}\right)^{\frac{p}{s_1}}\right\|_{\frac{q_1}{p}} \left\|\left(|\bar{g}|^{s_2} * |\tilde{\bar{\psi}}|^{s_2}\right)^{\frac{p}{s_2}}\right\|_{\frac{q_2}{p}} \\ &= \left\||f|^{s_1} * |\tilde{\phi}|^{s_1}\right\|^{\frac{p}{q_1}}_{\frac{q_1}{s_1}} \left\||\bar{g}|^{s_2} * |\tilde{\bar{\psi}}|^{s_2}\right\|^{\frac{p}{q_2}}_{\frac{q_2}{s_2}}.\end{aligned}$$

Hence, from Remark 4.1, it follows

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \left|V_\psi g\overline{V_\phi f}(x, \omega)\right|^p dw\right) dx \\ &\quad\leqslant C^{dp}_{s_1}C^{dp}_{s_2}\int_{\mathbb{R}^d}\left(|f|^{s_1} * |\tilde{\phi}|^{s_1}\right)^{\frac{p}{s_1}} \cdot \left(|\bar{g}|^{s_2} * |\tilde{\bar{\psi}}|^{s_2}\right)^{\frac{p}{s_2}}(x)\, dx \\ &\quad\leqslant C^{dp}_{s_1}C^{dp}_{s_2}\left(\left\||f|^{s_1} * |\tilde{\phi}|^{s_1}\right\|^{\frac{1}{s_1}}_{\frac{q_1}{s_1}} \left\||\bar{g}|^{s_2} * |\tilde{\bar{\psi}}|^{s_2}\right\|^{\frac{1}{s_2}}_{\frac{q_2}{s_2}}\right)^p \\ &\quad\leqslant K\left(\left\||f|^{s_1}\right\|^{\frac{1}{s_1}}_{\frac{p_1}{s_1}} \left\||g|^{s_2}\right\|^{\frac{1}{s_2}}_{\frac{p_2}{s_2}} \left\||\psi|^{s_2}\right\|^{\frac{1}{s_2}}_{\frac{p_2'}{s_2}} \left\||\tilde{\phi}|^{s_1}\right\|^{\frac{1}{s_1}}_{\frac{p_1'}{s_1}}\right)^p, \qquad (4.2)\end{aligned}$$

where $K = C_{s_1}^{dp} C_{s_2}^{dp} (C_{\frac{p_1}{s_1}} C_{\frac{p_1'}{s_1}} C_{(\frac{q_1}{s_1})'})^{\frac{dp}{s_1}} (C_{\frac{p_2}{s_2}} C_{\frac{p_2'}{s_2}} C_{(\frac{q_2}{s_2})'})^{\frac{dp}{s_2}}$. However, for $h \in L^r(\mathbb{R}^d)$, $\| |h|^s \|_{\frac{r}{s}}^{\frac{1}{s}} = (\int_{\mathbb{R}^d} |h|^{s \cdot \frac{r}{s}})^{\frac{s}{sr}} = \|f\|_r$. Inserting now, these into (4.2), yields

$$\iint_{\mathbb{R}^{2d}} \left| V_\psi g \overline{V_\phi f}(x,\omega)\right|^p dx\, d\omega \leqslant K \left(\|f\|_{p_1} \|g\|_{p_2} \|\phi\|_{p_1'} \|\psi\|_{p_2'} \right)^p.$$

Some computations show that

$$C_{s_j} C_{(\frac{q_j}{s_j})'}^{\frac{1}{s_j}} = q_j^{\frac{q_j-2}{2q_j}} (q_j-1)^{-\frac{1}{q_j}} (q_j-2)^{-\frac{q_j-2}{2q_j}}, \quad s_j = (q_j)',$$

and

$$C_{\frac{p_j}{s_j}}^{\frac{1}{s_j}} C_{\frac{p_j'}{s_j}}^{\frac{1}{s_j}} = (p_j-1)^{-\frac{p_j-1}{2p_j}} p_j^{\frac{1}{q_j}} (q_j-1)^{\frac{1}{q_j}} q_j^{-\frac{1}{2}} \left(q_j(p_j-1)-p_j\right)^{\frac{q_j(p_j-1)-p_j}{2p_jq_j}} \times (q_j-p_j)^{\frac{q_j-p_j}{2p_jq_j}},$$

that gives us

$$C_{s_j} C_{(\frac{q_j}{s_j})'}^{\frac{1}{s_j}} C_{\frac{p_j}{s_j}}^{\frac{1}{s_j}} C_{\frac{p_j'}{s_j}}^{\frac{1}{s_j}} = Q_j P_j,$$

and $K = (\prod_{j=1}^2 Q_j P_j)^{dp}$ as in the statement (b). □

Corollary 4.1. *Let us fix p_j, p_j', $j = 1,2$, with $\frac{1}{p_j} + \frac{1}{p_j'} = 1$. If $f \in L^{p_1}$, $\phi \in L^{p_1'}$, $g \in L^{p_2}$, $\psi \in L^{p_2'}$ and $h \leqslant p < \infty$, for $h = \frac{1}{2}\max\{p_1, p_1', p_2, p_2'\}$, then*

$$\iint_{\mathbb{R}^{2d}} \left| V_\psi g \overline{V_\phi f}(x,\omega)\right|^p dx\, dw \leqslant \left(Q^2 P_1 P_2\right)^{dp} \left(\|f\|_{p_1} \|g\|_{p_2} \|\phi\|_{p_1'} \|\psi\|_{p_2'} \right)^p,$$

where

$$Q = Q_1 = Q_2 = \frac{1}{\sqrt{2}} p^{-\frac{1}{2p}} (p-1)^{\frac{1-p}{2p}}$$

and P_1, P_2 are of the form

$$P_j = (p_j-1)^{-\frac{p_j-1}{2p_j}} p_j^{\frac{1}{2p}} \left(2p(p_j-1)-p_j\right)^{\frac{2p(p_j-1)-p_j}{4p_jp}} (2p-p_j)^{\frac{2p-p_j}{4p_jp}}, \quad j = 1,2.$$

Proof. It suffices to select $q_1 = q_2 = 2p$ in Theorem 4.1 □

Corollary 4.2. *If $f, \phi, g, \psi \in L^2(\mathbb{R}^d)$ and $1 \leqslant p < \infty$ then*

$$\iint_{\mathbb{R}^{2d}} \left| V_\psi g \overline{V_\phi f}(x,\omega)\right|^p dx\, dw \leqslant \left(\frac{1}{p}\right)^d \left(\|f\|_2 \|g\|_2 \|\phi\|_2 \|\psi\|_2 \right)^p.$$

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Proof. Since $p_1 = p_2 = 2$, in Corollary 4.1 we have $h = 1$ and also follows that $P_1 = P_2 = P = 2^{\frac{1}{2p}}(2p-2)^{\frac{p-1}{2p}}$, giving us

$$Q^2 P_1 P_2 = (QP)^2 = \frac{1}{2} p^{-\frac{1}{p}} (p-1)^{\frac{1-p}{p}} 2^{\frac{1}{p}} (2p-2)^{\frac{p-1}{p}} = p^{-\frac{1}{p}}$$

that means $(QP)^{2dp} = (\frac{1}{p})^d$. □

Remark 4.3. If $f = g$ and $\phi = \psi$, we obtain

$$\iint\limits_{\mathbb{R}^{2d}} |V_\psi g|^{2p}\, dx\, dw \leqslant \left(\frac{1}{p}\right)^d \left(\|f\|_2^2 \|\phi\|_2^2\right)^p, \quad p \geqslant 1,$$

which for $2p = q$ give us the well-known classical Lieb's inequality for the STFT $V_\psi g$:

$$\iint\limits_{\mathbb{R}^{2d}} |V_\psi g|^q\, dx\, dw \leqslant \left(\frac{2}{q}\right)^d \left(\|f\|_2 \|\phi\|_2\right)^q, \quad q \geqslant 2.$$

Let us now to prove the uncertainty principle of the generalized spectrogram:

Theorem 4.2. *Let $f \in L^{p_1}$, $\phi \in L^{p_1'}$, $g \in L^{p_2}$, $\psi \in L^{p_2'}$. If $U \subseteq \mathbb{R}^{2d}$ and $\varepsilon \geqslant 0$ are such that*

$$\iint\limits_{U} |V_\psi g \overline{V_\phi f}|\, dx\, dw \geqslant (1-\varepsilon) \|f\|_{p_1} \|g\|_{p_2} \|\phi\|_{p_1'} \|\psi\|_{p_2'},$$

then

$$\mu(U) \geqslant \sup_{p>h} (1-\varepsilon)^{\frac{p}{p-1}} \left(\prod_{j=1}^{2} Q_j P_j\right)^{\frac{dp}{1-p}}$$

Proof. Let p, p' be conjugate index, with $p > h$, $h = \frac{\max\{p_1, p_1'\} \max\{p_2, p_2'\}}{\max\{p_1, p_1'\} + \max\{p_2, p_2'\}}$. By Hölder inequality applied to the functions $V_\psi g \overline{V_\phi f} \in L^p$ and to the characteristic function on U, $\chi_U(x, \omega) \in L^{p'}$, we have

$$\begin{aligned} &(1-\varepsilon) \|f\|_{p_1} \|g\|_{p_2} \|\phi\|_{p_1'} \|\psi\|_{p_2'} \\ &\quad \leqslant \iint\limits_{U} |V_\psi g \overline{V_\phi f}|\, dx\, dw \\ &\quad \leqslant \left(\iint\limits_{\mathbb{R}^{2d}} |V_\psi g \overline{V_\phi f}|^p\, dx\, dw\right)^{\frac{1}{p}} \left(\iint\limits_{\mathbb{R}^{2d}} \chi_U(x, \omega)^{p'}\, dx\, dw\right)^{\frac{1}{p'}}. \end{aligned}$$

Now from Theorem 4.1 we conclude that

$$(1-\varepsilon) \leqslant \left(\prod_{j=1}^{2} Q_j P_j\right)^d \mu(U)^{\frac{p-1}{p}},$$

that is

$$\mu(U) \geqslant \sup_{p>h} (1-\varepsilon)^{\frac{p}{p-1}} \left(\prod_{j=1}^{2} Q_j P_j \right)^{\frac{dp}{1-p}}. \qquad \square$$

Remark 4.4. For $f, \phi, g, \psi \in L^2(\mathbb{R}^d)$ and $1 \leqslant p < \infty$ it follows that

$$\mu(U) \geqslant \sup_{p>1} (1-\varepsilon)^{\frac{p}{p-1}} \left(\frac{1}{p} \right)^{\frac{d}{1-p}},$$

and if $f = g$, $\phi = \psi$, we obtain the Lieb's uncertainty principle for the spectrogram

$$\mu(U) \geqslant \sup_{p>2} (1-\varepsilon)^{\frac{p}{p-2}} \left(\frac{2}{p} \right)^{\frac{2d}{2-p}}.$$

5. Characterization of the continuity in L^p spaces of localization operators

In this section we study the continuity in $L^q(\mathbb{R}^d)$ spaces of the localization operators $L^a_{\phi,\psi}$ defined by (2.8), where the symbol a is assumed to be in $L^p(\mathbb{R}^{2d})$. The continuity of localization operators in Lebesgue spaces has been studied in [4] for symbols belonging to $L^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant 2$; in [3] is treated also the case $p \geqslant 2$, proving that for $a \in L^p(\mathbb{R}^{2d})$ the operator

$$L^a_{\phi,\psi} : L^q(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d) \tag{5.1}$$

is continuous for every

$$q \in \left[\frac{2p}{p+1}, \frac{2p}{p-1} \right], \tag{5.2}$$

at least when the windows ϕ and ψ are in some suitable L^r spaces (in particular when they are in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$). A counterexample presented in [3] shows that there exist a symbol $a \in L^\infty(\mathbb{R}^{2d})$ and two windows ϕ, ψ such that the corresponding operator

$$L^a_{\phi,\psi} : L^\infty(\mathbb{R}^d) \longrightarrow L^\infty(\mathbb{R}^d)$$

is not continuous. Consequently, since $(L^{\bar{a}}_{\psi,\phi})^* = L^a_{\phi,\psi}$, the operator

$$L^{\bar{a}}_{\psi,\phi} : L^1(\mathbb{R}^d) \longrightarrow L^1(\mathbb{R}^d)$$

is not continuous, too. These results leave open the question of the continuity in Lebesgue spaces of localization operators in the remaining cases. We show in this section that the positive result proved in [3] is optimal, in the sense that for every p and q that do not satisfy (5.2) there exist a symbol $a \in L^p(\mathbb{R}^{2d})$ and two windows ϕ, ψ belonging to suitable $L^r(\mathbb{R}^d)$ spaces such that (5.1) is not continuous. In order to do this, we need the following result.

Proposition 5.1. *Let E, E_1, E_2, E_3 and E_4 be Banach spaces, and let E_2 be reflexive.*

(a) *Let us suppose that*

$$\sigma : E_3 \times E_4 \times E_2^* \times E_1 \longrightarrow E^*, \tag{5.3}$$

$\sigma = \sigma(\psi, \phi, g, f)$, is linear with respect to g and ϕ, antilinear with respect to f and ψ, and continuous. Then there exists a unique continuous map

$$(a, \phi, \psi) \in E \times E_4 \times E_3 \longmapsto T_{a,\phi,\psi} \in B(E_1, E_2), \tag{5.4}$$

linear with respect to a and ψ, and antilinear with respect to ϕ such that for every $g \in E_2^*$ we have

$$(g, T_{a,\phi,\psi} f) = \big(\sigma(\psi, \phi, g, f), a\big). \tag{5.5}$$

(b) *On the opposite direction, let the map* (5.4) *be continuous, linear with respect to a and ψ, and antilinear with respect to ϕ; then there exists a unique application*

$$\sigma : E_3 \times E_4 \times E_2^* \times E_1 \longrightarrow E^*, \tag{5.6}$$

linear with respect to g and ϕ, antilinear with respect to f and ψ, and continuous, such that (5.5) *holds for every $g \in E_2^*$.*

Proposition 5.1 was proved in a less general form in [1]. The proof in this case (that we present for completeness) is similar to the one in [1].

Proof. (a) Let us consider, for fixed ϕ, ψ, f and a, $a \in E$, the antilinear functional

$$g \in E_2^* \longrightarrow \overline{\big(\sigma(\psi, \phi, g, f), a\big)} \in \mathbb{C}. \tag{5.7}$$

By the continuity of (5.3) we have that

$$\left|\overline{\big(\sigma(\psi, \phi, g, f), a\big)}\right| \leqslant C_1 \|a\|_E \big\|\sigma(\psi, \phi, g, f)\big\|_{E^*} \leqslant C\|a\|_E \|\psi\|_{E_3} \|\phi\|_{E_4} \|g\|_{E_2^*} \|f\|_{E_1};$$

we then have that (5.7) belongs to E_2^{**}, that coincides with E_2, since E_2 is reflexive. Then there exists a unique $w = w(a, f, \phi, \psi) \in E_2$ such that

$$\big(\sigma(\psi, \phi, g, f), a\big) = (g, w).$$

Let us set $T_{a,\phi,\psi} f := w$; we then have (5.5). Let us prove now the continuity of the map (5.4): since $g \in E_2^*$ in (5.5) is arbitrary, we have that

$$\|T_{a,\phi,\psi} f\|_{E_2} \leqslant C\|a\|_E \|\phi\|_{E_4} \|\psi\|_{E_3} \|f\|_{E_1},$$

which implies that

$$\|T_{a,\phi,\psi}\|_{B(E_1,E_2)} \leqslant C\|a\|_E \|\phi\|_{E_4} \|\psi\|_{E_3}.$$

Then the map (5.4) is continuous.

(b) From (5.5) we immediately have that σ is linear with respect to g and ϕ, and antilinear with respect to f and ψ. As for the continuity of σ we observe that, from the continuity of (5.4),

$$\begin{aligned}\left|\big(\sigma(\psi, \phi, g, f), a\big)\right| &\leqslant C_1 \|g\|_{E_2^*} \|T_{a,\phi,\psi} f\|_{E_2} \\ &\leqslant C_2 \|g\|_{E_2^*} \|T_{a,\phi,\psi}\|_{B(E_1,E_2)} \|f\|_{E_1} \\ &\leqslant C_3 \|g\|_{E_2^*} \|a\|_E \|\phi\|_{E_4} \|\psi\|_{E_3} \|f\|_{E_1}.\end{aligned}$$

Since $a \in E$ is arbitrary, we then have

$$\big\|\sigma(\psi, \phi, g, f)\big\|_{E^*} \leqslant C_3 \|g\|_{E_2^*} \|\phi\|_{E_4} \|\psi\|_{E_3} \|f\|_{E_1},$$

that proves the continuity of (5.6). □

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We now pass to the analysis of the boundedness of the localization operator (2.8). We start by showing the continuity of (5.1) in an alternative way with respect to the proof in [3].

Theorem 5.1. *For every $a \in L^p(\mathbb{R}^{2d})$, $\phi \in L^{q'}(\mathbb{R}^d)$, $\psi \in L^q(\mathbb{R}^d)$ and $1 < q < \infty$ satisfying* (5.2) *the operator* (5.1) *is bounded.*

Proof. Let us observe that the following result holds. For every q and $\tilde{p}$, with $(2\tilde{p})' \leqslant q \leqslant 2\tilde{p}$, the map

$$V_\psi g \overline{V_\phi f} : (\psi, \phi, g, f) \in L^q(\mathbb{R}^d) \times L^{q'}(\mathbb{R}^d) \times L^{q'}(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \longmapsto V_\psi g \overline{V_\phi f} \in L^{\tilde{p}}(\mathbb{R}^{2d}) \tag{5.8}$$

is bounded; this is the result of Theorem 4.1 in the particular case $p_1 = p_2' = q$, where we have written $\tilde{p}$ in place of p just for convenience in the computations below. The relation (2.9) tells us that we are now in the situation of Proposition 5.1, where $V_\psi g \overline{V_\phi f}$ plays the role of $\sigma(\psi, \phi, g, f)$, $L^a_{\phi,\psi}$ plays the role of $T_{a,\phi,\psi}$, $E_1 = E_2 = E_3 = L^q(\mathbb{R}^d)$, $E_4 = L^{q'}(\mathbb{R}^d)$ and $E = L^{\tilde{p}'}(\mathbb{R}^{2d})$. Then Proposition 5.1(a), ensures us that the map

$$(a, \phi, \psi) \in L^{\tilde{p}'}(\mathbb{R}^{2d}) \times L^{q'}(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \longmapsto L^a_{\phi,\psi} \in B\big(L^q(\mathbb{R}^d), L^q(\mathbb{R}^d)\big)$$

is continuous. In particular, writing p in place of $\tilde{p}'$ (that means $\tilde{p} = p'$), we have that for every symbol $a \in L^p(\mathbb{R}^{2d})$ and for every windows $\phi \in L^{q'}(\mathbb{R}^d)$, $\psi \in L^q(\mathbb{R}^d)$ the corresponding localization operator $L^a_{\phi,\psi}$ is bounded on $L^q(\mathbb{R}^d)$, provided $(2p')' \leqslant q \leqslant 2p'$, i.e.

$$\frac{2p}{p+1} \leqslant q \leqslant \frac{2p}{p-1}.$$

The proof is then complete. □

Remark 5.1. The result proved in [3] is slightly more general than the one of Theorem 5.1, since [3] includes also the cases $q = 1$ and $q = +\infty$; the proof presented here does not work in these cases, since the space $E_2 = L^q(\mathbb{R}^d)$ in Proposition 5.1 is required to be reflexive. We recall anyway that for $q = 1$ or $q = +\infty$ the continuity of (5.1) can be proved directly by simple computations, cf. [3, Theorem 2.4].

We now want to prove a noncontinuity result for the localization operator $L^a_{\phi,\psi}$; in order to do this we need the following proposition.

Proposition 5.2. *Let us fix $q, r, \tilde{p} \in [1, \infty]$ in such a way that if $q \geqslant 2$, then $r \geqslant 2$, too, and vice versa if $q \leqslant 2$, then $r \leqslant 2$. We have that the map*

$$V_\psi g \overline{V_\phi f} : (\psi, \phi, g, f) \in L^r(\mathbb{R}^d) \times L^{r'}(\mathbb{R}^d) \times L^{q'}(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \longmapsto V_\psi g \overline{V_\phi f} \in L^{\tilde{p}}(\mathbb{R}^{2d}) \tag{5.9}$$

is not bounded for any r, q, $\tilde{p}$ satisfying

$$\frac{1}{\max\{r, r'\}} + \frac{1}{\max\{q, q'\}} < \frac{1}{\tilde{p}}. \tag{5.10}$$

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Proof. Let us fix, for $x \in \mathbb{R}^d$,

$$h(x) = e^{-\pi x^2}, \qquad h_\lambda(x) = e^{-\pi\lambda x^2}, \quad \lambda > 0, \tag{5.11}$$

and let $s, \tilde{s} \in [1, \infty]$. We recall from [1] that

$$\frac{\|V_h h_\lambda\|_{L^{\tilde{s}}}}{\|h\|_{L^{s'}}\|h_\lambda\|_{L^s}} = \frac{(s')^{\frac{d}{2s'}} s^{\frac{d}{2s}}}{\tilde{s}^{\frac{d}{\tilde{s}}}} \frac{(\lambda+1)^{\frac{d}{\tilde{s}}}}{(\lambda+1)^{\frac{d}{2}}} \lambda^{\frac{d}{2}(\frac{1}{s} - \frac{1}{\tilde{s}})}. \tag{5.12}$$

We want now to find a sequence $(\phi_\lambda, \psi_\lambda, g_\lambda, f_\lambda)$ such that the quantity

$$\frac{\|V_{\psi_\lambda} g_\lambda \overline{V_{\phi_\lambda} f_\lambda}\|_{L^{\tilde{p}}}}{\|\phi_\lambda\|_{L^{r'}}\|\psi_\lambda\|_{L^r}\|f_\lambda\|_{L^q}\|g_\lambda\|_{L^{q'}}}$$

is not bounded. We consider at first the case $q \geqslant 2$ (which implies $r \geqslant 2$); let us choose

$$(\psi_\lambda, \phi_\lambda, g_\lambda, f_\lambda) = (h_\lambda, h, h, h_\lambda), \tag{5.13}$$

h and h_λ being given by (5.11). We observe that, since h and h_λ are real-valued, we have

$$V_{h_\lambda} h(x, \omega) = e^{-2\pi i x\omega} V_{\bar{h}} \bar{h}_\lambda(-x, \omega) = e^{-2\pi i x\omega} V_h h_\lambda(-x, \omega);$$

since $|e^{-2\pi i x\omega}| = 1$, we then have

$$\left\| V_{h_\lambda} h(x,\omega) \overline{V_h h_\lambda}(x,\omega)\right\|_{L^{\tilde{p}}} = \left\| V_h h_\lambda(-x,\omega) \overline{V_h h_\lambda}(x,\omega)\right\|_{L^{\tilde{p}}}. \tag{5.14}$$

Now, $V_h h_\lambda$ can be computed explicitly, cf. [1], obtaining

$$V_h h_\lambda(x,\omega) = (\lambda+1)^{-\frac{d}{2}} e^{2\pi i \frac{1}{\lambda+1} x\omega} h_{\frac{\lambda}{\lambda+1}}(x) g_{\frac{1}{\lambda+1}}(\omega);$$

so, since $h_{\frac{\lambda}{\lambda+1}}(x)$ is even, we can replace $V_h h_\lambda(-x,\omega)$ in (5.14) by $V_h h_\lambda(x,\omega)$, without changing the norm. By these last observations and (5.12) we then obtain

$$\begin{aligned}
\frac{\|V_{h_\lambda} h \overline{V_h h_\lambda}\|_{L^{\tilde{p}}}}{\|h\|_{L^{r'}}\|h_\lambda\|_{L^r}\|h_\lambda\|_{L^q}\|h\|_{L^{q'}}} &= \frac{\|V_h h_\lambda \overline{V_h h_\lambda}\|_{L^{\tilde{p}}}}{\|h\|_{L^{r'}}\|h_\lambda\|_{L^r}\|h_\lambda\|_{L^q}\|h\|_{L^{q'}}} \\
&= \frac{\|V_h h_\lambda\|_{L^{2\tilde{p}}}}{\|h\|_{L^{r'}}\|h_\lambda\|_{L^r}} \frac{\|V_h h_\lambda\|_{L^{2\tilde{p}}}}{\|h_\lambda\|_{L^q}\|h\|_{L^{q'}}} \\
&= \frac{(r')^{\frac{d}{2r'}} r^{\frac{d}{2r}} (q')^{\frac{d}{2q'}} q^{\frac{d}{2q}}}{(2\tilde{p})^{\frac{d}{\tilde{p}}}} \frac{(\lambda+1)^{\frac{d}{\tilde{p}}}}{(\lambda+1)^d} \lambda^{\frac{d}{2}(\frac{1}{r} + \frac{1}{q} - \frac{1}{\tilde{p}})},
\end{aligned}$$

this last expression tends to $+\infty$ for $\lambda \to 0^+$, as we can deduce from (5.10). Then the map (5.9) is not bounded.

Till now we have considered the case $q \geqslant 2$; if $q \leqslant 2$ (which implies $r \leqslant 2$) we choose

$$(\psi_\lambda, \phi_\lambda, g_\lambda, f_\lambda) = (h, h_\lambda, h_\lambda, h)$$

in place of (5.13) and we repeat the same calculations as before. □

We can now prove the following noncontinuity result for localization operators.

Theorem 5.2. *Let us consider the localization operator $L^a_{\phi,\psi}$, and let $p, q \in [1, \infty]$, $p \neq 1$, such that*

$$q < \frac{2p}{p+1} \quad or \quad q > \frac{2p}{p-1}, \tag{5.15}$$

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where we mean $\frac{2p}{p+1} = \frac{2p}{p-1} = 2$ *for* $p = +\infty$. *Then there exist a symbol* $a \in L^p(\mathbb{R}^{2d})$ *and two windows* $\phi \in L^{r'}(\mathbb{R}^d)$, $\psi \in L^r(\mathbb{R}^d)$,

$$r = \begin{cases} 2p' & \text{if } q > \frac{2p}{p+1}, \\ (2p')' & \text{if } q < \frac{2p}{p-1} \end{cases} \tag{5.16}$$

such that

$$L^a_{\phi,\psi} \notin B\big(L^q\big(\mathbb{R}^d\big)\big), \tag{5.17}$$

where we have written $B(L^q(\mathbb{R}^d))$ *for* $B(L^q(\mathbb{R}^d), L^q(\mathbb{R}^d))$.

Proof. (i) We consider at first the case when $q \neq 1$ and $q \neq +\infty$; we want to use, as in the proof of Theorem 5.1, the general result of Proposition 5.1: a comparison between (2.9) and (5.5) shows that we are allowed to use Proposition 5.1 in the particular case $\sigma(\psi, \phi, g, f) = V_\psi g \overline{V_\phi f}$ and $T_{a,\phi,\psi} = L^a_{\phi,\psi}$. Then the negative result of Proposition 5.2, together with Proposition 5.1(b), applied for $E = L^{\tilde{p}'}(\mathbb{R}^{2d})$, $E_1 = E_2 = L^q(\mathbb{R}^d)$ (E_2 is reflexive, since $q \neq 1$, $q \neq +\infty$), $E_3 = L^r(\mathbb{R}^d)$, $E_4 = L^{r'}(\mathbb{R}^d)$, ensures us that the map

$$(a, \phi, \psi) \in L^{\tilde{p}'}\big(\mathbb{R}^{2d}\big) \times L^{r'}\big(\mathbb{R}^d\big) \times L^r\big(\mathbb{R}^d\big) \longmapsto L^a_{\phi,\psi} \in B\big(L^q\big(\mathbb{R}^d\big)\big)$$

is not continuous for any $\tilde{p}$, q and r such that

$$\frac{1}{\max\{r, r'\}} + \frac{1}{\max\{q, q'\}} < \frac{1}{\tilde{p}},$$

where r and q are supposed to satisfy the hypotheses of Proposition 5.2. Let us write for simplicity p instead of $\tilde{p}'$ (which implies $\tilde{p} = p'$), and fix r such that

$$\max\{r, r'\} = 2p' \tag{5.18}$$

(observe that (5.18) is equivalent to (5.16) because of the hypotheses of Proposition 5.2); we then have that the map

$$(a, \phi, \psi) \in L^p\big(\mathbb{R}^{2d}\big) \times L^{r'}\big(\mathbb{R}^d\big) \times L^r\big(\mathbb{R}^d\big) \longmapsto L^a_{\phi,\psi} \in B\big(L^q\big(\mathbb{R}^d\big)\big) \tag{5.19}$$

is not continuous for any p and q satisfying

$$\frac{1}{\max\{q, q'\}} < \frac{1}{2p'}.$$

Observe that this last condition is equivalent to (5.15), and so we have that the map (5.19) is not continuous for any p and q satisfying (5.15).

We want now to prove that (5.19) is not everywhere defined, i.e. there exist a symbol a and two windows ϕ and ψ in the corresponding spaces such that the localization operator $L^a_{\phi,\psi}$ is not bounded on $L^q(\mathbb{R}^d)$. To this aim it is enough to prove that the graph of the map (5.19) is closed, and then the Closed Graph Theorem ensures us that the map itself is not everywhere defined. Let us take a sequence

$$(a_j, \phi_j, \psi_j) \longrightarrow (a, \phi, \psi) \quad \text{in } L^p\big(\mathbb{R}^{2d}\big) \times L^{r'}\big(\mathbb{R}^d\big) \times L^r\big(\mathbb{R}^d\big),$$

such that the corresponding localization operators

$$L^{a_j}_{\phi_j,\psi_j} \longrightarrow A \quad \text{in } B\big(L^q\big(\mathbb{R}^d\big)\big); \tag{5.20}$$

we have to prove that

$$A = L^a_{\phi,\psi}. \tag{5.21}$$

We shall show that for every $u, v \in \mathcal{S}(\mathbb{R}^d)$,

$$(Au, v) = \big(L^a_{\phi,\psi} u, v\big). \tag{5.22}$$

Now, from (5.20) we get

$$\big(L^{a_j}_{\phi_j,\psi_j} u, v\big) \longrightarrow (Au, v). \tag{5.23}$$

On the other hand, the continuity of (5.8) in the particular case $q = r'$ and $\tilde{p} = p'$, ensures us that

$$V_{\psi_j} u \overline{V_{\phi_j} v} \longrightarrow V_\psi u \overline{V_\phi v} \quad \text{in } L^{p'}\big(\mathbb{R}^{2d}\big), \tag{5.24}$$

since r satisfies (5.18); then, from (2.9), (5.24) and since $a_j \to a$ in $L^p(\mathbb{R}^{2d})$ we get

$$\big(L^{a_j}_{\phi_j,\psi_j} u, v\big) = \big(a_j, V_{\psi_j} v \overline{V_{\phi_j} u}\big) \longrightarrow \big(a, V_\psi v \overline{V_\phi u}\big) = \big(L^a_{\phi,\psi} u, v\big). \tag{5.25}$$

Now, comparing (5.23) with (5.25) we immediately get (5.22). Then, (5.21) is a consequence of (5.22) and standard density arguments.

(ii) Let us consider now $q = +\infty$. We want to prove that for every $p \in (1, +\infty]$ there exist a symbol $a \in L^p(\mathbb{R}^{2d})$ and two windows $\phi \in L^{(2p')'}(\mathbb{R}^d)$, $\psi \in L^{2p'}(\mathbb{R}^d)$ such that

$$L^a_{\phi,\psi} \notin B\big(L^\infty\big(\mathbb{R}^d\big)\big).$$

Let us suppose an absurd that for every $(a, \phi, \psi) \in L^p(\mathbb{R}^{2d}) \times L^{(2p')'}(\mathbb{R}^d) \times L^{2p'}(\mathbb{R}^d)$ the corresponding localization operator

$$L^a_{\phi,\psi} \in B\big(L^\infty\big(\mathbb{R}^d\big)\big). \tag{5.26}$$

Now, Theorem 5.1, applied for $q = \frac{2p}{p-1} = 2p'$, ensures us that for every $(a, \phi, \psi) \in L^p(\mathbb{R}^{2d}) \times L^{(2p')'}(\mathbb{R}^d) \times L^{2p'}(\mathbb{R}^d)$,

$$L^a_{\phi,\psi} \in B\big(L^{2p'}\big(\mathbb{R}^d\big)\big). \tag{5.27}$$

Then by (5.26), (5.27) and interpolation theory we obtain that $L^a_{\phi,\psi} \in B(L^q(\mathbb{R}^d))$ for every $q \in [\frac{2p}{p-1}, +\infty]$ and for every (a, ϕ, ψ) in the corresponding spaces; this last fact contradicts what we have proved at the point (i), and so the conclusion holds also for $q = +\infty$.

(iii) The remaining case $q = 1$ can be treated in the same way as $q = +\infty$. More directly, since we have found $(a, \phi, \psi) \in L^p(\mathbb{R}^{2d}) \times L^{(2p')'}(\mathbb{R}^d) \times L^{2p'}(\mathbb{R}^d)$ such that $L^a_{\phi,\psi} \notin B(L^\infty(\mathbb{R}^d))$, from the relation $(L^{\bar{a}}_{\psi,\phi})^* = L^a_{\phi,\psi}$ we immediately obtain that

$$L^{\bar{a}}_{\psi,\phi} \notin B\big(L^1\big(\mathbb{R}^d\big)\big),$$

and then the proof is complete. □

Summarizing the results of Theorems 5.1 and 5.2, we have proved that, considering the localization operator $L^a_{\phi,\psi}$, cf. (2.8), with symbol $a \in L^p(\mathbb{R}^{2d})$ (and windows in the corresponding spaces), we can ensure its continuity as an operator $L^a_{\phi,\psi} : L^q(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ if and only if

$$q \in \left[\frac{2p}{p+1}, \frac{2p}{p-1}\right],$$

in the sense that when $q \notin [\frac{2p}{p+1}, \frac{2p}{p-1}]$ we can find suitable windows ϕ and ψ that make $L^a_{\phi,\psi}$ not continuous on $L^q(\mathbb{R}^d)$.

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