Gevrey hypoellipticity of p-powers of non-hypoelliptic operators

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Abstract. We characterize the hypoellipticity in C^∞ and Gevrey G^λ classes of 2-variable PDO's conteining powers of anisotropic principal terms. We use an approach based on methods from microlocal analysis. Conditions are imposed on the coefficients of lower order terms. Also a semilinear version is proposed considering C^∞ nonlinear perturbations, see Theorem 1.1 and Theorem 1.6.

1. Introduction

This paper deals with the properties of hypoellipticity and Gevrey hypoellipticity of a class of partial differential operators with multiple characteristics conteining p-powers of anisotropic principal terms, i.e. the derivatives with highest anisotropic order; in two variables $z = (x, y)$, belonging to the set Ω, neighborhood of a point $z_0 = (x_0, y_0) \in \mathbb{R}^2$. In particular we consider equations of the form:

$$P(x, y, D_x, D_y)u + F(x, y, D_x^l D_y^j u)\Big|_{\frac{m}{d}l+j<k^*} = 0, \tag{1.1}$$

where the linear part is given by:

$$P(x, y, D_x, D_y) = Q^p + \sum_{k^* \le \frac{m}{d}l+j<pm} b_{lj}(x, y) D_x^l D_y^j \tag{1.2}$$

and the operator Q is of the form:

$$Q(x, y, D_x, D_y) = D_y^m - a(x, y) D_x^d, \tag{1.3}$$

with $m, d, j, l, p \in \mathbb{Z}_+$, $p \ge 2$, $0 < k^* < pm$, $D_x = -i\frac{\partial}{\partial x}$, $D_y = -i\frac{\partial}{\partial y}$; the anisotropic principal terms are represented by the operator Q. We shall also say that $\frac{m}{d}l + j$ is the anisotropic order of $D_x^l D_y^j$, so the nonlinearity involves derivatives of anisotropic order less than k^*. For related results in the case $p = 1$, see De Donno-Oliaro [5], where assuming $\operatorname{Re} a(x, y) \ne 0$ and looking for a couple (l^*, j^*) with $k^* := \frac{m}{d}l^* + j^* > m - \frac{1}{2}$, such that $\operatorname{Im} b_{l^*j^*}(x, y) \ne 0$, a sufficient condition for hypoellipticity with loss of the smoothness of solutions equal to $m - k^*$ has been estabilished for (1.1) and (1.2). On the coefficients $\operatorname{Im} b_{lj}(x, y)$ of the terms having

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anisotropic order larger than k^*, we assumed to be of the same sign of $\mathrm{Im}\, b_{l^*j^*}(x,y)$ (allowing they can vanish); the same assumption on $-\mathrm{Im}\, a(x,y)$. Also local solvability was showed for (1.1) and (1.2); and G^λ hypoellipticity, $\lambda \geq \frac{1}{k^*-(m-1)}$, for (1.1). As standard, the Gevrey space $G^\lambda(\Omega)$, is defined by the estimates:

$$f \in C^\infty(\Omega), \quad \sup_K |\partial_z^\alpha f(z)| \leq C_K^{|\alpha|+1}(\alpha!)^\lambda, \text{ for every compact } K \subset\subset \Omega,$$

with $\lambda \geq 1$.

In the case $\mathrm{Im}\, b_{l^*j^*}(x,y)$ vanishes at some points (but not identically), Gevrey solvability for small Gevrey index was proved, too.

In De Donno-Oliaro [5] in spite of the conditions imposed on the lower order terms are optimal (the results are sharp in the particular case of constant coefficients), we just obtained a small hypoelliptic interval $(m-\frac{1}{2}, m)$, in which we look for a couple (l^*, j^*) in order to impose our hypotheses (the interval is $(m-1, m)$ in the case of constant coefficients). For operators (costant coefficients) containing p-powers of anisotropic principal terms of the form

$$(D_y^2 - D_x)^p + iD_x^{l^*} D_y^{j^*} \quad , \tag{1.4}$$

k^* can runs in the interval $(p, 2p)$, which is $p-1$ anisotropic orders in width, see De Donno-Oliaro-Rodino [6].

In view to enlarge the hypoelliptic interval of the variable coefficients case, according with the model (1.4) of constant coefficients, we consider general operators in the form (1.2) having p-powers of the operator Q. Unfortunaly the influence of higher order terms of Q^p, $p \geq 4$, limit k^* in the interval $(pm - 1 - \frac{1}{2}, pm)$, that anyway contains the anisotropic order $pm - \frac{1}{2}$ not accepted in the case $p = 1$ ($k^* > m - \frac{1}{2}$). For $p \leq 3$, the influence of (the higher order terms of) Q^p is irrelevant, since $pm - 1 - \frac{1}{2} \leq p(m - \frac{1}{2})$, and $p(m-\frac{1}{2})$ is the best limitation from the below we can obtain with our techniques. Let us also observe that in the worst case we get $k^* \in (pm - \frac{1}{2} - \frac{1}{d}, pm) \supset (pm - \frac{1}{2}, pm)$, see Remark 1.3.

On the contrary in the generic differential polynomial symbols

$$(\eta^m - a(x,y)\xi^d)^p + \sum_{(l,j)\in I} c_{lj}(x,y)\xi^l\eta^j \ , \tag{1.5}$$

where $(\xi,\eta) \in \mathbb{R}^2$ are the dual variables of (x,y); conteining as particular case the symbol of the operators (1.2), k^* belonging in the interval $\big(p(m-\frac{1}{2}), pm\big)$ should be accepted for any integer p; see Theorem 2.1. The subset of indeces I corresponds to lower order terms.

In this paper we propose a complete analysis of the influence of the lower-order terms of (1.5) on the hypoellipticity of (1.1) and (1.2) in the C^∞ category and in the Gevrey spaces G^λ beyond the critical index $m/(m-1)$. The arguments in our proofs are based mainly on microlocal tools: pseudo-differential operators, wave front sets, allowing relevant simplifications in the study, and $S^m_{\rho,\delta}$ techniques. In the present paper, for simplicity, we consider real value function $a(x,y)$ ($\mathrm{Im}\, a(x,y) \equiv 0$)

in (1.3). We suppose C^∞ coefficients in (1.2), (1.3), and in the follow we always ask that:

$$d < m, \quad a(x_0, y_0) \neq 0, \tag{1.6}$$

$$F(z;t) = \sum_{r \in \mathbb{Z}_+^M} C_r(z) t^r, \; C_r \in C^\infty(\Omega), \; t \in \mathbb{Z}^M, \tag{1.7}$$

where, for every compact $K \subset\subset \Omega$, $\sup_{z \in K} |D^\alpha C_r(z)| \leq C_{\alpha,K} \lambda_r$ and moreover $\tilde{F}(t) = \sum_r \lambda_r t^r$ is entire analytic.

The nonzero requirement on $a(x,y)$ in (1.6) is an invariant nondegeneracy condition, usually required in the study of the hypoellipticity (and local solvability) of the linear operator (1.2) with $p = 1$, in the C^∞ and Gevrey G^λ, $\lambda > \frac{m}{m-1}$, see for example, Liess-Rodino[16], De Donno-Rodino[7], where Gevrey hypoellipticity for PDE's with high multiplicity is proved in the isotropic case. For $p = 1$ see also Gramchev-Popivanov[13], where hypotheses are given on the roots of the full symbol of the operator, and Corli[3], about results of non-hypoellipticity. Let us also observe that if $\operatorname{Im} a(x,y) \neq 0$ then the operator is quasi-elliptic; the results on hypoellipticity (and local solvability) are well known in this case.

A new important element respect to the case $p = 1$, for formulating our results in this paper, is the so called k^{max} order, i.e. the higest anisotropic order $\frac{m}{d}l + j$, $k^* < \frac{m}{d}l + j < pm$ of $D_x^l D_y^j$ such that the corresponding coefficient is not identically null. When $k^* \geq \frac{m}{d}l + j$ for all couples (l,j) we say that does not exists k^{max}.

We split the symbol of (1.2) in the following way:

$$\begin{aligned} p(x,y,\xi,\eta) = (\eta^m - a(x,y)\xi^d)^p + \sum_{k^* \leq \frac{m}{d}l + j \leq k^{max}} \tilde{b}_{lj}(x,y)\xi^l \eta^j \\ + \sum_{\frac{m}{d}l + j < k^*} \tilde{b}_{lj}(x,y) D_x^l D_y^j \end{aligned} \tag{1.8}$$

where $\tilde{b}_{lj}(x,y) = b_{lj}(x,y)$, for $\frac{m}{d}l + j > pm - 1$ since the operator Q^p captures derivatives of anisotropic order less or equal than $pm-1$, and $\tilde{b}_{lj}(x,y) = a_{xy}^{rs}(x,y) + \operatorname{Re} b_{lj}(x,y) + i\operatorname{Im} b_{lj}(x,y)$, for $m \leq k^* \leq \frac{m}{d}l + j \leq pm - 1$, with r, s representing suitable derivative orders respectively with respect x and y variables. We are not interessed to the coefficients of mixed derivatives having anisotropic order less than k^*.

Moreover we define the anisotropic characteristic manifold

$$\Sigma := \{(x,y,\xi,\eta) \in \Omega \times \mathbb{R}^2 \setminus \{0\}, \; (\eta^m - a(x,y)\xi^d)^p = 0\}. \tag{1.9}$$

Let us state the main results.

Theorem 1.1. *Let $(l^*, j^*) \in \mathbb{Z}_+^2$ be the unique couple having anisotropic order $k^* := \frac{m}{d}l^* + j^*$, with $\max\{p(m - \frac{1}{2}), k^{max} - \frac{1}{2}\} < k^* < pm$. We suppose $a(x,y)$,*

$b_{lj}(x,y) \in C^\infty(\Omega)$ and assume that for $(x,y,\xi,\eta) \in \Sigma$ the following conditions hold:

i) $\operatorname{Im} b_{l^*j^*}(x,y) \neq 0$,

ii) $\operatorname{Im} b_{l^*j^*}(x,y)\operatorname{Im} b_{lj}(x,y)\xi^{l+l^*}\eta^{j+j^*} \geq 0,$ for all $(l,j),$ such that $k^* < \frac{m}{d}l + j \leq k^{max}$

iii) $\operatorname{Re}\tilde{b}_{lj}(x,y)\xi^l\eta^j \geq 0,$ for all $(l,j),$ such that $k^* \leq \frac{m}{d}l + j \leq k^{max}$
iii′) for odd p : $\operatorname{Re}\tilde{b}_{lj}(x,y) \equiv 0$ for all (l,j) s.t. $k^* \leq \frac{m}{d}l + j \leq k^{max}$.

Assume moreover that (1.6) *holds. Then the operator P in* (1.2) *is C^∞-hypoelliptic with loss of regularity equal to $pm - k^*$ derivatives. Moreover, taking analytic coefficients a, b_{lj},* (1.2) *is G^λ-hypoelliptic for*

$$\lambda \geq \begin{cases} \frac{p}{k^*-p(m-1)} & , \quad k^{max} \leq \frac{p-1}{p}k^* + m \\ \frac{1}{k^*-(k^{max}-1)} & , \quad k^{max} > \frac{p-1}{p}k^* + m \end{cases} . \tag{1.10}$$

Remark 1.2. It is always possible to rephrase the previous assumptions in Theorem 1.1 directly on the coefficients of P. For example, if $a(x,y) > 0$ and m, d are odd, the conditions $i)$, $ii)$, $iii)$ are respectively equivalent to: $i')$ $\operatorname{Im} b_{l^*j^*}(x,y) > 0$ ($<$ 0); $ii')$ for all (l,j) such that $\frac{d}{m}l + j > k^*$, $\operatorname{Im} b_{lj}(x,y) \geq 0\,(\leq 0)$ for $j + j^*$ and $l + l^*$ both even or both odd and $\operatorname{Im} b_{lj}(x,y) \equiv 0$ otherwise; $iii')$ $\operatorname{Re}\tilde{b}_{lj}(x,y) \geq 0)$ for j and l both even or both odd and $\operatorname{Re}\tilde{b}_{lj}(x,y) \equiv 0$ otherwise.

Remark 1.3. In the operators (1.2), the order k^{max} is always larger or equal than $pm-1$. Assuming that it is exactly $pm-1$ (just the influence of Q^p), the condition $k^* > \max\{p(m-\frac{1}{2}), k^{max} - \frac{1}{2}\}$ gives us the sharp limitation from the below $p(m-\frac{1}{2})$ whenever $pm - 1 - \frac{1}{2} \leq p(m - \frac{1}{2})$, that is $p \leq 3$. Otherwise, allowing k^{max} to be the largest order less than pm, $k^{max} = pm - \frac{1}{d}$, we again obtain the best limitation $p(m-\frac{1}{2})$ if and only if $p \leq 1 + \frac{2}{d}$, that represents the following conditions: $\{d = 1$ and $p = 1, 2\}$ or $\{d = 2$ and $p = 2\}$. Observe that $k^* > pm - \frac{1}{d} - \frac{1}{2}$ for $k^* > k^{max} - \frac{1}{2}$, in this case. In general for $k^{max} = pm - \frac{h}{d}$, for $1 \leq h \leq d$, the best limitation is obtained for $p \leq 1 + \frac{2h}{d}$. Let us also observe that the step from an anisotropic order and the next one in intervals of the type $(pm-1, pm)$ is exactly $\frac{1}{d}$, since we have $dpm - d < ml + dj < dpm$.

Remark 1.4. If $a(x,y)$ in the operator Q in (1.3) is a constant function, in Theorem 1.1 we can consider $k^* > p(m-1)$ if k^{max} there is not, $k^* > \max\{\frac{p}{p+1}(k^{max} + m - 1), k^{max} - \frac{1}{2}\}$ otherwise. It follows directly by the proof of the Theorem 2.1 in the next section, see Remark 2.11.

Remark 1.5. Th hypoellipticity results in G^λ for P in Theorem1.1 imply local Gevrey G^λ solvability of the adjoint operator P^*. This follows from the main theorem in Albanese-Corli-Rodino [1]

As example of operator with odd p we consider the following:

Example.

$$(D_y^3 - (y^2+1)D_x^2)^3 + (18y+i)D_x^2 D_y^5 \tag{1.11}$$

where we have not k^{max}, since there are not anisotropic orders strictly larger than $k^* = \frac{3}{2}2 + 5 = 8 > 3(3-\frac{1}{2})$. Of course hypothesis $i)$ in 1.1 is satisfied. The symbol of (1.11) is

$$(\eta^3 - (y^2+1)\xi^2)^3 - (18y)D_x^2 D_y^5 + (18y+i)D_x^2 D_y^5 + \text{lower o. ts.}$$

so $\mathrm{Re}\,\tilde{b}_{l^*j^*}(x,y) \equiv 0$. We have hypoellipticity and G^λ hypoellipticity for $\lambda \geq \frac{3}{8-3(3-1)} = \frac{3}{2}$. More general we can consider operators of the form

$$(D_y^m - (y^2+1)D_x^d)^3 + (6my+i)D_x^d D_y^{2m-1} \quad ,$$

hypoelliptic and G^λ hypoelliptic for $\lambda \geq \frac{3}{2}$.

As relevant example with constant function $a(x,y)$ we can consider p-powers of the Schrödinger operator:

Example. for $p=2$

$$(D_y^2 - D_x)^2 + i(y^2+1)D_x D_y \quad ,$$

is hypoelliptic and G^λ hypoelliptic for $\lambda \geq 2$, see also [6] for the isotropic frame. Instead for the operator

$$(D_y^2 - D_x)^3 + i(y^2+1)D_x D_y^2 + D_x D_y^3 \quad ,$$

nothing we can conclude about its hypoellipticity since the term $k^{max} = 5$ is too high. In general operators of the form

$$(D_y^2 - D_x)^p + i(y^2+1)D_x^l D_y^{p+1-2l} \quad ,$$

are hypoelliptic and G^λ hypoelliptic for $\lambda \geq 2$.

As full example of operator satisfying Theorem 1.1 we consider:

Example.

$$(D_y^6 - (y^2+1)D_x^5)^2 + (y^4+i)D_x^6 D_y^4 + x^2 D_x^8 D_y^2 \quad ,$$

where $k^{max} = \frac{58}{5}$, $k^* = \frac{56}{5} > k^{max} - \frac{1}{2} > 2(6-\frac{1}{2})$; i), ii), iii) in Theorem 1.1 hold, so we have hypoellipticity and G^λ hypoellipticity for $\lambda \geq \frac{5}{3}$, since $k^{max} = \frac{p-1}{p}k^* + m$, see (1.10).

In the follow the Sobolev anisotropic space $H^s_{(\frac{d}{m},1)}(\mathbb{R}^2)$ will be used, defined by:

$$\|f\|^2_{H^s_{(\frac{d}{m},1)}} := \int (1 + |\xi|^{2\frac{d}{m}} + |\eta|^2)^s \, |\widehat{f}(\xi,\eta)|^2 \, d\xi \, d\eta \ < \infty, \tag{1.12}$$

$\widehat{f}(\xi,\eta)$ being the Fourier transform of $f(x,y)$. For $s > \frac{1+\frac{m}{d}}{2}$, $H^s_{(\frac{d}{m},1)}$ is an algebra, cf. the inhomogeneous Schauder estimates in Garello [9, Proposition 2.5].

Theorem 1.6. *Under the assumptions of Theorem 1.1 on P and the condition* (1.7) *on F, let u be a solution of* (1.1) *which belongs to $H^s_{(\frac{d}{m},1),\mathrm{loc}}(\Omega)$, for $s \geq s_0$, where s_0 is a sufficiently large fixed real number. Then $u \in C^\infty(\Omega)$.*

Let us observe that several results have been written for microlocal regularity for nonlinear elliptic PDEs with analytic nonlinearities as in (1.7), cf., Lascar[10], see also Popivanov-Iordanov[20].

About the critical bound from below for s, we recall that the regularity statement of Theorem1.6 is not true if the solution u belongs to H^s with $s > 0$ but small enough. For explicit constructions of strongly singular solutions in H^s_{loc} $s < \frac{n}{2}$ of semilinear elliptic PDE, cf. Biagioni-Gramchev [2] and Gramchev [11].

In the next Section 2 we prove Theorem 1.1 using $S^m_{\rho,\delta}$ estimates; Theorem 1.6 is proved in Section 3.

2. Hypoellipticity for the class (1.5) of differential polynomials

In this section we prove $S^m_{\rho,\delta}$ estimates of the following class of differential polynomial in suitable simplectic co-ordinates:

$$p(x,y,\xi,\eta) = (\eta^m - a(x,y)\xi^d)^p + \sum_{(l,j)\in I} c_{lj}(x,y)\xi^l\eta^j \tag{2.1}$$

Where $m, d, j, l, p \in \mathbb{Z}_+$, $d < m$, $p \geq 2$ even, and $a : \Omega \to \mathbb{R}$, $c_{lj} : \Omega \to \mathbb{C}$ smooth functions. As before we denote by $z = (x, y)$ the real variables in Ω, open subset in $\mathbb{R}^2$, neighborhood of a point z_0; $\zeta = (\xi, \eta)$ the dual variables of z. We define the following sets for $k \in \mathbb{Q}_+$, $0 < k < pm$:

$$I_k = \{ (l,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : \frac{m}{d} l + j = k \}$$

and fix $k = k^*$ such that $p(m - \frac{1}{2}) < k^* < pm$. We use the notations k^- for all $k < k^*$, k^+ for all $k > k^*$ and k^{max} for the biggest value k^+. We define $I = I_- \cup I_{k^*} \cup I_+$, with $I_- = \cup I_{k^-}$, $I_+ = \cup I_{k^+}$.

We consider the anisotropic norm $\lambda(\zeta) = |\xi|^{\frac{d}{m}} + |\eta|$, cf. the expression of the anisotropic Sobolev spaces $H^s_{(\frac{d}{m},1)}(\mathbb{R}^2)$, in (1.12). We recall that $\Sigma := \{(z,\zeta) \in \Omega \times \mathbb{R}^2 \setminus \{0\} : (\eta^m - a(x,y)\xi^d)^p = 0\}$ is the anisotropic characteristic manifold of $p(x, y, \xi, \eta)$ in (2.1); let Λ be a neighborhood of Σ, we denote with Γ the set $\Omega \times \Lambda$, and we state the following:

Theorem 2.1. *Assume that I_{k^*} consists of just one couple (l^*, j^*), $k^* = \frac{m}{d} l^* + j^*$ with $\max\{p(m-\frac{1}{2}), k^{max} - \frac{1}{2}\} < k^* < pm$ such that:*

i) $\mathrm{Im}\, c_{l^* j^*}(x,y) \neq 0$, for all $(x,y) \in \Omega$;

ii) for all (l,j), s.t. $k^* < \frac{m}{d} l + j \leq k^{max}$,
$\mathrm{Im}\, c_{l^* j^*}(x,y) \mathrm{Im}\, c_{lj}(x,y) \xi^{l+l^*} \eta^{j+j^*} \geq 0$, $(z,\zeta) \in \Gamma$;

iii) $\mathrm{Re}\, c_{lj}(x,y) \xi^l \eta^j \geq 0$, for all (l,j) s.t. $k^* \leq \frac{m}{d} l + j \leq k^{max}$;

iv) $a(x,y) \neq 0$, forall $(x,y) \in \Omega$;

Then

$$|p(x,y,\xi,\eta)| \geq b\lambda(\zeta)^{k^*} \quad in\ \Omega \times \mathbb{R}^2,$$

for a suitable constant $b > 0$; and for all $(\alpha,\beta) \in \mathbb{Z}^2_+$, $(\gamma,\theta) \in \mathbb{Z}^2_+$ and for all $K \subset\subset \Omega$ we have with suitable constants $L_{\alpha,\beta,\gamma,\delta}$ and B that:

(2.2)
$$\frac{|D_x^\alpha D_y^\beta D_\xi^\gamma D_\eta^\theta p(x,y,\xi,\eta)|\, \lambda(\zeta)^{\rho(\gamma \frac{m}{d} + \theta) - \delta(\alpha \frac{m}{d} + \beta)}}{|p(x,y,\xi,\eta)|} \leq L_{\alpha,\beta,\gamma,\theta}\ ,\ |\xi| + |\eta| > B,$$

with

(2.3) $$\rho = \min\{\frac{k^* - p(m-1)}{p}, k^* - k^{max} + 1\}\ ,\ \delta = \max\{\frac{pm - k^*}{p}, k^{max} - k^*\}.$$

Remark 2.2. Observe that it is always $\delta < \rho$ since

$$\rho = \begin{cases} \frac{k^* - p(m-1)}{p} & , \quad k^{max} \leq \frac{p-1}{p} k^* + m \\ k^* - k^{max} + 1 & , \quad k^{max} > \frac{p-1}{p} k^* + m \end{cases},$$

$$\delta = \begin{cases} \frac{pm - k^*}{p} & , \quad k^{max} \leq \frac{p-1}{p} k^* + m \\ k^{max} - k^* & , \quad k^{max} > \frac{p-1}{p} k^* + m \end{cases},$$

and we have assumed $k^* > \max\{p(m-\frac{1}{2}), k^{max} - \frac{1}{2}\}$.

Remark 2.3. By formula (2.2) and by Mascarello-Rodino([18], Theorem 3.3.6), we have that an operator $P(x,y,D_x,D_y)$, associated to the symbol $p(x,y,\xi,\eta)$ in (2.1), is C^∞-microlocally hypoelliptic in Γ; i.e. $\Gamma \cap WF\, Pu = \Gamma \cap WF\, u$, for all $u \in \mathcal{D}'(\Omega)$, where $WF\, u$ is the Hörmander wave front set. A microhypoelliptic operator is hypoelliptic too.

Remark 2.4. If the coefficients are analytic, formula (2.2) holds for $L_{\alpha\beta\gamma\theta} = L^{\alpha+\beta+\gamma+\theta+1} \alpha! \beta! \gamma! \theta!$, so by Kajitani-Wakabayashi([17], Theorem 1.9), we have that an operator $P(z,D)$, associated to the symbol $p(z,\zeta)$ in (2.1) is G^λ-microlocally hypoelliptic in Γ for $\lambda \geq \max\left\{\frac{1}{\rho}, \frac{1}{1-\delta}\right\} = \frac{1}{\rho}$ (or equivalently $\frac{1}{1-\delta}$), that is $\lambda \geq \frac{p}{k^* - p(m-1)}$ for $k^{max} \leq \frac{p-1}{p} k^* + m$ and $\lambda \geq \frac{1}{k^* - (k^{max} - 1)}$ for $k^{max} > \frac{p-1}{p} k^* + m$.

Remark 2.5. When $\rho < 1$, and $\delta > 0$, one can prove by means of interpolation theory as in Wakabayashi([23] Theorem 2.6) that (2.2) is valid for any $(\alpha, \beta, \gamma, \theta) \in \mathbb{Z}_+^4$, if (2.2) holds for $\alpha + \beta + \gamma + \theta = 1$. Hence it is sufficient to verify (2.2) for $\alpha+\beta+\gamma+\theta = 1$ since the relations $k^* < k^{max} < pm$ give the inequalities required.

Remark 2.6. For the proof of the Theorem 1.1 it will be sufficient to apply Theorem 2.1. We recall that in the operators of the form (1.2) always is $k^{max} \geq pm - 1$.

Proof of Theorem 2.1. First we estimate the numerator of (2.2) and then we give some lemmas to estimate the denominator, see Lemma 2.7, Lemma 2.9, Lemma 2.10. If $\alpha = 1$, we get

$$\begin{aligned} &|D_x p(x, y, \zeta)|\, \lambda(\zeta)^{-\delta \frac{m}{d}} = \\ &\quad |p(\eta^m - a(x, y)\xi^d)^{p-1} D_x a(x, y)\xi^d + \sum_{(l,j)\,\in\, I} D_x c_{lj}(x, y)\, \xi^l \eta^j\, |\lambda(\zeta)^{-\delta \frac{m}{d}} \\ &\quad \leq L_1(|\eta^m - a(x, y)\xi^d|^{p-1}|\xi|^d + \sum_{(l,j)\,\in\, I} |\xi|^l|\eta|^j|\ )\lambda(\zeta)^{-\delta \frac{m}{d}}; \end{aligned}$$

and similarly for $\beta = 1$

$$|D_y p(x, y, \zeta)|\, \lambda(\zeta)^{-\delta} \leq L_2(|\eta^m - a(x, y)\xi^d|^{p-1}|\xi|^d + \sum_{(l,j)\,\in\, I} |\xi|^l|\eta|^j|\ )\lambda(\zeta)^{-\delta},$$

for suitable constants $L_1,\ L_2$. If $\gamma = 1$,

$$|D_\xi\, p(z, \xi, \eta)|\lambda(\zeta)^{\rho \frac{m}{d}} \leq L_3(|\eta^m - a(x, y)\xi^d|^{p-1}|\xi|^{d-1} + \sum_{(l,j)\,\in\, I} |\xi|^{l-1}|\eta|^j\ )\lambda(\zeta)^{\rho \frac{m}{d}};$$

and for $\theta = 1$

$$|D_\eta\, p(z, \xi, \eta)|\lambda(\zeta)^\rho \leq L_4(|\eta^m - a(x, y)\xi^d|^{p-1}|\eta|^{m-1} + \sum_{(l,j)\,\in\, I} |\xi|^l|\eta|^{j-1}\ )\lambda(\zeta)^\rho,$$

with suitable constants L_3, L_4.

To prove (2.2), it will be then sufficient to show the boundedness, for $|\zeta| > B$, of the functions

$$\begin{aligned} Q_1(z, \zeta) &= \frac{\left(|\eta^m - a(x, y)\xi^d|^{p-1}|\xi|^d + \sum_{(l,j)\,\in\, I} |\xi|^l|\eta|^j|\right)\lambda(\zeta)^{-\delta}}{|p(z, \zeta)|}, \\ Q_2(z, \zeta) &= \frac{\left(|\eta^m - a(x, y)\xi^d|^{p-1}|\eta|^{m-1} + \sum_{(l,j)\,\in\, I} |\xi|^l|\eta|^{j-1}\right)\lambda(\zeta)^\rho}{|p(z, \zeta)|}, \\ Q_3(z, \zeta) &= \frac{\left(|\eta^m - a(x, y)\xi^d|^{p-1}|\xi|^{d-1} + \sum_{(l,j)\,\in\, I} |\xi|^{l-1}|\eta|^j\right)\lambda(\zeta)^{\rho \frac{m}{d}}}{|p(z, \zeta)|}. \end{aligned}$$

First we introduce three regions:

$$\begin{array}{ll} R_1: & c\,|\xi|^d \leq |\eta|^m \leq C\,|\xi|^d \\ R_2: & |\eta|^m \geq C\,|\xi|^d \\ R_3: & |\eta|^m \leq c\,|\xi|^d \end{array}, \tag{2.4}$$

for suitable constants c, C satisfying for every compact set K in Ω the inequalities $c < \frac{1}{2}\min_{(x,y)\in K} |a(x,y)|$, and $C > 2\max_{(x,y)\in K} |a(x,y)|$, cf. [4], [5], [7], [8]. We understand the neighborhood Λ to be the region R_1.

The following estimates then hold:

$$\lambda(\zeta)^{-\delta} \leq \begin{cases} const.\,|\eta|^{-\delta} & , \quad \zeta \in R_1 \qquad (I) \\ const.|\eta|^{-\delta} & , \quad \zeta \in R_2 \qquad (II) \\ const.|\xi|^{-\delta\frac{d}{m}} & , \quad \zeta \in R_3\,; \quad (III) \end{cases} \tag{2.5}$$

and

$$\lambda(\zeta)^{\rho} \leq \begin{cases} const.\,|\eta|^{\rho} & , \quad \zeta \in R_1 \\ const.\,|\eta|^{\rho} & , \quad \zeta \in R_2 \\ const.\,|\xi|^{\rho\frac{d}{m}} & , \quad \zeta \in R_3\,; \end{cases} \tag{2.6}$$

note that (II) and (III) in (2.5) hold for all $\zeta \in \mathbb{R}^2$ but for our aim we may limit ourselves to consider them respectively in R_2 and in R_3. By abuse of notation, in the following we shall also denote by R_1, R_2, R_3 the sets $\Omega\times R_1, \Omega\times R_2, \Omega\times R_3$; recall that $\Gamma = \Omega\times\Lambda$.

The following three lemmas give us some relevant estimates from the below of $|p(x,y,\xi,\eta)|$ in (2.1).

Lemma 2.7. *Let $p(z,\zeta)$ be the function* (2.1)*, such that* (i),(ii) *and* (iii) *in Theorem 2.1 hold. Then there are positive constants $K_1 < 1$, B, such that, for $(z,\zeta) \in R_1$, $|\zeta| > B$:*

$$|p(z,\zeta)| \geq K_1 \left((\eta^m - a(x,y)\xi^d)^{2p} + |\mathrm{Im}\, c_{l^*j^*}(x,y)|^2 |\xi|^{2l^*} |\eta|^{2j^*}\right)^{\frac{1}{2}}. \tag{2.7}$$

Proof. We have that

$$\begin{aligned} |p(z,\zeta)|^2 &= \left((\eta^m - a(x,y)\,\xi^d)^p + \textstyle\sum_{(l,j)\in I} \mathrm{Re}\, c_{lj}(x,y)\,\xi^l\eta^j\right)^2 \\ &+ \Big(\mathrm{Im}\, c_{l^*j^*}(x,y)\xi^{l^*}\eta^{j^*} + \textstyle\sum_{(l,j)\in I_+} \mathrm{Im}\, c_{lj}(x,y)\,\xi^l\eta^j \\ &+ \textstyle\sum_{(l,j)\in I_-} \mathrm{Im}\, c_{lj}(x,y)\,\xi^l\eta^j\Big)^2 . \end{aligned} \tag{2.8}$$

By developing (2.8) and removing the terms

$$\Big(\sum_{(l,j)\in I} \mathrm{Re}\, c_{lj}(x,y)\xi^l\eta^j\Big)^2 \quad , \quad \Big(\sum_{(l,j)\in I_+} \mathrm{Im}\, c_{lj}(x,y)\xi^l\eta^j + \sum_{(l,j)\in I_-} \mathrm{Im}\, c_{lj}(x,y)\xi^l\eta^j\Big)^2$$

respectively from the real and imaginary part of $p(z,\zeta)$, we can write

$$|p(z,\zeta)|^2 \geq (\eta^m - a(x,y)\,\xi^d)^{2p} + \mathrm{Im}\, c_{l^*j^*}(x,y)^2\,\xi^{2l^*}\eta^{2j^*} + \sum_{i=1}^{4} J_i(z,\zeta)$$

where

$$J_1(z,\zeta) = 2\mathrm{Im}\, c_{l^*j^*}(x,y) \sum_{(l,j)\in I_+} \mathrm{Im}\, c_{lj}(x,y)\,\xi^{l^*+l}\eta^{j^*+j}\,, \tag{2.9}$$

$$J_2(z,\zeta) \;=\; 2(\eta^m \;-\; a(x,y)\,\xi^d)^p \sum_{(l,j)\in I_{k^*}\cup I_+} \mathrm{Re}\, c_{lj}(x,y)\,\xi^l\eta^j \tag{2.10}$$

$$J_3(z,\zeta) \;=\; 2\mathrm{Im}\, c_{l^*j^*}(x,y) \sum_{(l,j)\in I_-} \mathrm{Im}\, c_{lj}(x,y)\,\xi^{l^*+l}\eta^{j^*+j}\,, \tag{2.11}$$

$$J_4(z,\zeta) \;=\; 2(\eta^m \;-\; a(x,y)\,\xi^d)^p \sum_{(l,j)\in I_-} \mathrm{Re}\, c_{lj}(x,y)\,\xi^l\eta^j \tag{2.12}$$

The functions are non-negative, (2.9) and (2.10) are also non-negative respectively by hypotheses (ii) and (iii), since p is even. Let us fix attention on $J_3(z,\zeta)$ and $J_4(z,\zeta)$ defined respectively by (2.11) and (2.12). We have for all $\epsilon > 0$

$$\frac{1}{2}(\mathrm{Im}\, c_{l^*j^*}(x,y))^2\,\eta^{2j^*}\xi^{2l^*} + J_3(z,\zeta) \ge (\frac{1}{2}-\epsilon)(\mathrm{Im}\, c_{l^*j^*}(x,y))^2\,\eta^{2j^*}\xi^{2l^*}\,,$$

in R_1, $|\zeta| > B$. In fact by (2.4) in R_1 and hypothesis (i) in Theorem 2.1, for all $\epsilon > 0$ we get for B sufficiently large

$$\frac{|J_3(z,\zeta)|}{(\mathrm{Im}\, c_{l^*j^*}(x,y))^2\,\eta^{2j^*}\xi^{2l^*}} \;\le\; const \sum_{(l,j)\in I_-} \frac{|\xi|^{l^*+l}|\eta|^{j^*+j}}{\xi^{2l^*}\eta^{2j^*}} \;\le$$

$$\le\; const \sum_{(l,j)\in I_-} \frac{|\eta|^{j^*+j+(l^*+l)\frac{m}{d}}}{\eta^{2j^*+2l^*\frac{m}{d}}} \;<\; \epsilon\,, \quad |\zeta| > B.$$

We remark that $k^* \;=\; \frac{m}{d}l^* + j^* \;>\; k^- \;=\; \frac{m}{d}l + j$. Concerning (2.12) we also have that:

$$\frac{1}{2}(\eta^m \;-\; a(x,y)\,\xi^d)^{2p} + \frac{1}{2}\mathrm{Im}\, c_{l^*j^*}(x,y)^2\,\xi^{2l^*}\eta^{2j^*} + J_4(z,\zeta)$$
$$\ge (\frac{1}{2}-\varepsilon)[(\eta^m \;-\; a(x,y)\,\xi^d)^{2p} + \mathrm{Im}\, c_{l^*j^*}(x,y)^2\,\xi^{2l^*}\eta^{2j^*}]$$

since

$$\frac{|J_4(z,\zeta)|}{(\eta^m \;-\; a(x,y)\,\xi^d)^{2p} + \mathrm{Im}\, c_{l^*j^*}(x,y)^2\,\xi^{2l^*}\eta^{2j^*}} < \varepsilon,\;\; |\zeta| > B. \tag{2.13}$$

In fact, considering the region $\xi^d = (\frac{1}{a(x,y)} - r)\eta^m$ in the plane (ξ,η), with $(x,y) \in K \subset\subset \Omega$, $|r| < 1$; we obtain by (2.4) in R_1 that the left part of (2.13) is estimated by

$$\sum_{(l,j)\in I_-} \frac{r^p|\eta|^{mp+\frac{m}{d}l+j}}{r^{2p}|\eta|^{2mp} + |\eta|^{2k^*}} = \sum_{(l,j)\in I_-} \frac{t^p|\eta|^{mp+\frac{m}{d}l+j-2k^*}}{|\eta|^{2mp-2k^*} + t^{2p}} < \varepsilon, \quad |t| + |\eta| > B, \tag{2.14}$$

where $t = \frac{1}{r}$, by dividing for η^{2k^*}; since $k^- < k^*$. Recall that:

for all $\varepsilon > 0$ there exists $B_\varepsilon > 0$ such that for $|x| + |y| > B$,

$\dfrac{x^\alpha y^\beta}{x^{2\gamma} + x^{2\nu}} < \varepsilon$ if and only if $(2\gamma - \alpha)(2\nu - \beta) > \alpha\beta$.

For $|r| \geq 1$, in R_1, we have

$$\sum_{(l,j)\in I_-} \frac{t^p|\eta|^{mp+\frac{d}{m}l+j-2k^*}}{|\eta|^{2mp-2k^*}+t^{2p}} \leq |\eta|^{-mp+\frac{m}{d}l+j} < \varepsilon, \quad \text{since } k^- < mp. \tag{2.15}$$

Then

(2.16)

$$|p(z,\zeta)| \geq K_1 \left((\eta^m - a(x,y)\xi^d)^{2p} + |\mathrm{Im}\, c_{l^*j^*}(x,y)|^2|\xi|^{2l^*}|\eta|^{2j^*}\right)^{\frac{1}{2}}, \ |\zeta| > B.$$

for a suitable positive constant K_1. □

Remark 2.8. From the previous estimate (2.16) follows easily that $|p(z,\zeta)| \geq |\zeta|^{k^*}$ in R_1.

Lemma 2.9. *Let $p(z,\zeta)$ be the function* (2.1). *Then there are positive constants $K_2 < 1$, B, such that:*

$$|p(z,\zeta)| \geq K_2\,|\eta|^{pm}, \quad (z,\zeta) \in R_2,\ |\zeta| > B. \tag{2.17}$$

Lemma 2.10. *Let $p(z,\zeta)$ be the function* (2.1), *such that* (*iv*) *in* (2.1) *holds. Then there are positive constants $K_3 < 1$, B, such that:*

$$|p(z,\zeta)| \geq K_3\,|\xi|^{pd}, \quad (z,\zeta) \in R_3,\ |\zeta| > B. \tag{2.18}$$

For the proofs of Lemma 2.9 and Lemma 2.10, see De Donno-Rodino [7], concerning the case $p = 1$, the general case $p \geq 2$ actually does not involve more complications.

By the relations (2.4), (2.5), (2.6), and the inequality $|\eta^m - a(x,y)\xi^d| \leq |\eta|^m + C|\xi|^d$, we easily estimate the numerators N_i, $i = 1,2,3$, of Q_i, in the regions R_2 and R_3 in the following way:

$$N_1(\zeta) \leq \begin{cases} const.\,|\eta|^{pm-\delta} & , \ \zeta \in R_2 \\ const.\,|\xi|^{pd-\delta\frac{d}{m}} & , \ \zeta \in R_3; \end{cases} \tag{2.19}$$

$$N_2(\zeta) \leq \begin{cases} const.\,|\eta|^{pm-1+\rho} & , \ \zeta \in R_2 \\ const.\,|\xi|^{pd-\frac{d}{m}(1-\rho)} & , \ \zeta \in R_3; \end{cases} \tag{2.20}$$

$$N_3(\zeta) \leq \begin{cases} const.\,|\eta|^{pm-\frac{m}{d}(1-\rho)} & , \ \zeta \in R_2 \\ const.\,|\xi|^{pd-1+\rho} & , \ \zeta \in R_3. \end{cases} \tag{2.21}$$

We have $N_2(z,\zeta) \leq N_3(z,\zeta)$ in R_2 and R_3 since $d < m$ and $\rho < 1$; so we can just consider the functions $Q_1(z,\zeta)$ and $Q_2(z,\zeta)$ in those regions. Now Lemma 2.9, Lemma 2.10 and the estimates (2.19), (2.20) show the boundedness of $Q_1(z,\zeta)$ and $Q_2(z,\zeta)$ in the regions R_2 and R_3, so for $Q_3(z,\zeta)$, too.

Regarding the region R_1, we observe that $|\eta^m - a(x,y)\xi^d|$ vanishes in it, so estimates of the previous type are not optimal.

In R_1 by (2.5),(2.7) we get easily:

$$Q_1(z,\zeta) \le const \frac{|\eta^m - a(x,y)\xi^d|^{p-1}|\eta|^{m-\delta}}{((\eta^m - a(x,y)\xi^d)^{2p} + \operatorname{Im} c_{l^*j^*}(x,y)^2\xi^{2l^*}\eta^{2j^*})^{\frac{1}{2}}}$$

(2.22)

$$+ \sum_{(l,j)\in I_{max}} \frac{|\xi|^{\frac{m}{d}l+j-\delta}}{|\eta|^{k^*}} \le L,\ (x,y)\in K,\ |\zeta| > B,$$

where $I_{max} \subset I_+$, s.t. $\frac{m}{d}l + j = k^{max}$. The second term in right-hand side is bounded since $\delta \ge k^{max} - k^*$ in (2.3). About the first term we can argue in the same way we have done in Lemma 2.7, see formulas (2.13), (2.14), (2.15); since also $\delta \ge \frac{pm-k^*}{p}$ in (2.3).

Remark 2.11. If $a(x,y)$ is a constant function, the first part of the right-hand side in (2.22) completly vanishes (first derivatives with respect z), so we just consider $\delta \ge k^{max} - k^*$ for the boundedness of Q_1 in R_1. Imposing the fundamental inequality in the $S^m_{\rho,\delta}$ framework, $\delta < \rho$, with ρ in (2.3) we find the new conditions for k^*: $k^* > \max\{p(m-1), \frac{p}{p+1}(k^{max}+m-1), k^{max} - \frac{1}{2}\}$.

The study of the boundedness of the functions $Q_i(z,\zeta)$, $i = 2,3$ in the region R_1 actually does not involve further complicated statements, so arguing like the previous step and using the estimates (2.3) on ρ, i.e. $\rho = \min\{\frac{k^*-p(m-1)}{p}, k^* - k^{max} + 1\}$ we have proved that $Q_i(z,\zeta)$, $i = 1,2,3$ is also bounded in R_1.

Now the following remark ends the proof:

Remark 2.12. By formulas (2.7), (2.17), (2.18), we obtain that $|p(z,\zeta)| \ge a\,|\zeta|^{\frac{d}{m}k^*}$, $a > 0$, $|\zeta| > B$. In fact we obtain that $|p(z,\zeta)| \ge const\,|\eta|^{k^*}$ in R_1, then $|\eta|^{k^*} = \frac{1}{2}|\eta|^{k^*} + \frac{1}{2}|\eta|^{k^*} \ge const\,(|\eta|^{k^*} + |\xi|^{\frac{d}{m}k^*}) \ge const\,(|\eta|^{\frac{d}{m}k^*} + |\xi|^{\frac{d}{m}k^*}) \sim const\,|\zeta|^{\frac{d}{m}k^*}$, so $|p(x,\xi)| \ge a\,|\zeta|^{k^*\frac{d}{m}}$. In the same way we get $|p(x,\xi)| \ge a\,|\zeta|^{dp}$ in R_2 and R_3, the result follows . since we have $k^* < pm$. If we refer to the anisotropic weight function $\lambda(\zeta) = |\xi|^{\frac{d}{m}} + |\eta| \sim (|\xi|^d + |\eta|^m)^{\frac{1}{m}}$, we find, in the same way, that $|p(z,\zeta)| \ge b\lambda(\zeta)^{k^*}$ *in* Γ, for $|\xi| > B$, for a suitable positive constant $b > 0$.

□

The next Section3 is devoted to prove Theorem 1.6.

3. Sobolev estimates and semilinear version of the result

Remark 3.1. If we refer to the anisotropic weight function $\lambda(\zeta) = |\xi|^{\frac{d}{m}} + |\eta| \sim (|\xi|^d + |\eta|^m)^{\frac{1}{m}}$, we find, in the same way we have done in Remark 2.12, that $|p(z,\zeta)| \ge b\lambda(\zeta)^{k^*}$ *in* Γ, for $|\xi| > B$, for a suitable positive constant $b > 0$.

Let us assume now $p(z,\zeta)$ defined in all $\Omega \times \mathbb{R}^2$ and denote by P the corresponding operator. Theorem 1.1 and Remark 3.1 show respectively that the loss of regularity of P is $\varepsilon = pm - \frac{d}{m}k^*$ in the isotropic case and $\varepsilon^* = pm - k^*$ in the anisotropic case. Since $\varepsilon^* < \varepsilon$, it is natural to allow a semilinear version of the result of regularity of Theorem 1.1, that is expressed by Theorem 1.6 and that we prove in the sequel.

Let us before consider again the semilinear equation

$$P(x, y, D_x, D_y)u + F(x, y, D_x^l D_y^j u)_{\big| \frac{m}{d}l+j<k^*} = 0 \tag{3.1}$$

where $p(x, y, D_x, D_y)$ is the model considered in (2.1), such that the hypotheses of the Theorem 2.1 hold. Moreover, let us recall that F is of the type (1.7), i.e. $F(z;t) = \sum_{r\in\mathbb{Z}_+^M} C_r(z)t^r,\ C_r \in C^\infty(\Omega),\ t \in \mathbb{Z}^M$. Now we prove Theorem1.6:

*proof of Theorem*1.6. Observe that $D_x^l D_y^j u \in H^{s-(\frac{m}{d}l+j)}_{(\frac{d}{m},1),loc}(\Omega)$. Note that $\frac{m}{d}l + j$ actually implies $\frac{m}{d}l + j \leq k^* - \gamma$, with $\gamma > 0$. Then

$$p(x, y, D_x, D_y)u = -F(x, y, D_x^l D_y^j u)_{\big| \frac{m}{d}l+j<k^*} \in H^{s-k^*+\gamma}_{(\frac{d}{m},1),loc}(\Omega),$$

using here the assumption $s \geq s_0$, see Garello ([9], Remark 2.4). By the last part of Remark 2.12 we have that $u \in H^{s+\gamma}_{(\frac{d}{m},1),loc}(\Omega)$.

Using again Garello ([9], Remark 2.4) we get that $p(x, D)u \in H^{s-\frac{k^*}{q}+2\gamma}_{(\frac{d}{m},1),loc}(\Omega)$ and in its own turn $u \in H^{s+2\gamma}_{(\frac{d}{m},1),loc}(\Omega)$. Repeating now the preceding argument we obtain $u \in \cap_{t\in\mathbb{R}^+} H^t_{(\frac{d}{m},1),loc}(\Omega)$, that is $u \in C^\infty(\Omega)$.

□

Remark 3.2. It is possible to propose a generalization of Theorems 2.1 considering the complete influence of lower order terms of (2.1) in the the general case $a(x, y)$ complex value function, that leads to a more complicated situations; or also to operators with involutive characteristics of high multiplicity, in more than two space variables. It shall detailed in a future paper.

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