A class of quadratic time-frequency representations based on the short-time Fourier transform

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Abstract. Motivated by problems in signal analysis, we define a class of time-frequency representations which is based on the short-time Fourier transform and depends on two fixed windows. We show that this class can be viewed as a link between the classical Rihaczek representation and the spectrogram. Correspondingly we formulate for this class a suitable general form of the uncertainty principle which have, as limit case, the uncertainty principles for the Rihaczek representation and for the spectrogram. We finally consider the questions of marginal distributions. We compute them in terms of convolutions with the windows and prove simple conditions for which average and standard deviation of the distributions in our class coincide with that of their marginals.

1. Time-frequency Representations

The total energy of a signal $f(x)$, $(x \in \mathbb{R}^d)$, is commonly interpreted in physics and in signal analysis as its L^2 norm $\|f\|_{L^2}$. Accordingly, the positive function $|f(x)|^2$ represents the density of the energy of the signal with respect to the time (although x is the "time" variable we shall assume here $x \in \mathbb{R}^d$ for generality and also in view of possible other interpretations). As the Fourier transform $\hat{f}(\omega)$ gives the frequencies contained in the signal, then $|\hat{f}(\omega)|^2$ represents the energy density with respect to the frequency variable ω and the equality $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$, stated by the Plancherel theorem, is interpreted, in this context, as the natural assertion that the total energy of a signal does not change if we switch from the time representation $f(x)$ to the frequency representation $\hat{f}(\omega)$. Actually the sentence "the signal $f(x)$", which is of common use and we shall also adopt, is somehow misleading because $f(x)$ and $\hat{f}(\omega)$ should be viewed as two different, and in some sense complementary, representations of the same physical phenomenon constituted by the signal. In the L^2 frame, which corresponds to the space of finite energy signals, the fact that the Fourier transform is a bijection means that each of the two representations contain

Received by the editors December 15, 2005.
1991 *Mathematics Subject Classification.* Primary 47G30; Secondary 42B10, 47G10.
Key words and phrases. time-frequency representations, short-time Fourier transform, uncertainty principles, Rihaczek distribution.

in reality all informations about the signal. However in each representation some informations are contained in an explicit and others in a more implicit form and this can make the latter difficult to be recognized. For example it is clear that from the time representation $f(x)$ of a signal the support supp f immediately indicates when the signal takes place in time, however the frequencies contained in the signal are not to be detected directly but given only through the "form" of the graph of the function $f(x)$. In a quite symmetrical way the function $\hat{f}(\omega)$ contains explicit informations about the frequencies of the signal but the information about when these frequencies are present in the signal is hidden in the "form" of the function $\hat{f}(\omega)$. For example the Fourier transform $e^{2\pi ia\omega}\hat{f}(\omega)$ of the delayed signal $f(x-a)$ differs from the transform $\hat{f}(\omega)$ of the original signal $f(x)$ only in the "shape" factor $e^{2\pi ia\omega}$. A clear exposition on these subjects can be found for example in [13].

One fundamental problem of time-frequency analysis is to find suitable representations for the energy density of a signal simultaneously with respect to time and frequency. Of course this question is equivalent to the definition of a $2d-$dimensional distribution density $Q(f)(x,\omega)$ depending on the variables $x \in \mathbb{R}^d$ and $\omega \in \mathbb{R}^d$. However, differently to what happens with usual distributions in statistics and probability theory, here the uncertainty principle put intrinsic limitations to the natural properties one would expect from a $2d$-dimensional distribution. The basic problem is that "instantaneous frequency" has no physical meaning, and therefore, no matter which time-frequency representation $Q(f)$ is used, its point values do not have any univocal physical interpretation. perfectionated for particular time-frequency representations).

More precisely, let $Q(f)(x,\omega)$ be a time-frequency distribution of the energy of a signal f. Then some desirable features are expressed by the following conditions:

(1.1)
- Positivity: $Q(f)(x,\omega)$ should be real and positive.
- No spreading effect:
  If supp $f \subset I$ for an interval $I \subset \mathbb{R}^d$ then $\Pi_x$supp $Q(f) \subset I$ (where Π_x is the orthogonal projection from $\mathbb{R}^d_x \times \mathbb{R}^d_\omega$ to $\mathbb{R}^d_x$); analogously, if supp $\hat{f} \subset J$ then $\Pi_\omega$supp $Q(f) \subset J$.
- Marginal distributions:
  $\int_{R^d} Q(f)(x,\omega)dx = |\hat{f}(\omega)|^2$; $\int_{R^d} Q(f)(x,\omega)d\omega = |f(x)|^2$.

It turns out that these conditions are actually incompatible with the uncertainty principle and they can therefore be satisfied only with a certain degree of approximation.

Many different representations have been defined in the literature with the aim of approaching as near as possible an ideal representation (see [5], [8], [9], [12], [13]). A general frame for the study of these representations is given by the so-called *Cohen class*, see for reference [3], [4].

Three of the most known representations are the *spectrogram*, the *Rihaczek* and the *Wigner* representations.

For what concerns the spectrogram, the idea, originally due to D. Gabor, is to focus the Fourier transform on small intervals in time and to analyze the frequencies present in these intervals. This is performed by multiplying the signal $f(t)$ by a cut-off, or *window*, function $\phi(t)$ that can be translated by a parameter x along the time axis, before taking the Fourier transform. In this way we are lead to the definition of the *Gabor transform* or *short-time Fourier transform* (briefly *STFT*) of a signal $f(t)$:

$$V_\phi f(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi it\omega}\overline{\phi(t-x)}f(t)dt. \tag{1.2}$$

It indicates the frequencies ω contained in the signal in a neighborhood of the time x. The conjugation on the window appears just for mathematical convenience so that $V_\phi f(x,\omega) = (f, \phi_{x,\omega})_{L^2}$, where $\phi_{x,\omega}(t) = e^{2\pi i\omega t}\phi(t-x)$.
The quadratic representation corresponding to the STFT is called *spectrogram* (see [4], [10]) and is defined as

$$Sp_\phi(f)(x,\omega) = |V_\phi f(x,\omega)|^2. \tag{1.3}$$

It is of course a positive distribution but it does not satisfies the marginals and has a spreading effect depending on the support of the window ϕ.

The Rihaczek quadratic representation is essentially defined as the product of the signal $f(x)$ with its Fourier transform $\hat{f}(\omega)$, more precisely it is the distribution

$$R(f)(x,\omega) = e^{-2\pi ix\omega}f(x)\overline{\hat{f}(\omega)}. \tag{1.4}$$

Though its elementary definition it has reasonable physical motivations and was widely used in the time-frequency analysis of signals (see [4], [11]). As one can immediately verify, it satisfies the marginals and has no spreading effect, however it is evidently not positive.

The third very popular representation is the Wigner distribution (see [18]) defined as

$$Wig(f)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi ix\omega}f(x+t/2)\overline{f(x-t/2)}dt. \tag{1.5}$$

As the Rihaczek representation, it is not positive but it has no spreading effect and it satisfies the marginals (see [4], [10], [12]).

The Wigner distribution and the STFT are the basic tools for the theory of Weyl and localization operators respectively, about these wide fields there exists a huge literature, see for example [2], [16], [17], [19], [15], [20].

We shall be addressed in this paper essentially to the first two above-mentioned representations; related questions about the Wigner representation will be treated in a further paper. We shall establish a link between the spectrogram and the Rihaczek representation by defining in the next section a class of time-frequency representations $Q(f)(x, \omega) = Q_{\phi,\psi}(f)(x, \omega)$ which depend on two window functions or distributions ϕ, ψ. In a suitable general functional setting for windows and signals, we shall show that a sort of path between the spectrogram and the Rihaczek representation is allowed within this class, permitting to pass from one representation to the other by a continuous variation of the windows. Of course the degree of approximation by which the conditions (1.1) are satisfied will also change accordingly. As we explain in section 2, the motivation of our definition is not only a mathematical one but it has been suggested by the attempt to reduce the spreading effect of the spectrogram, result which can be easily confirmed by computer simulations.

In section 3 we treat the question of the uncertainty principle. We formulate a simple uncertainty principle for the representation $Q_{\phi,\psi}$ in the L^p frame and show that it links different form of uncertainty principles for the Rihaczek and spectrogram representations. Finally in section 4 we consider the problem of the marginal distributions. We compute the marginals of the representation $Q_{\phi,\psi}$ in terms of convolution with the product of the windows and deduce from this that the case where our representation reduces to the Rihaczek is the only case where both marginal conditions in (1.1) are satisfied. Further we give in the general case simple conditions on the windows in order that average and standard deviation of the marginals of $Q_{\phi,\psi}$ coincide with that of the "expected" marginal distributions $|f(x)|^2$ and $|\hat{f}(\omega)|^2$.

2. The Representation $Q_{\phi,\psi}$

Let $q : E \times E \to F$ be a sesquilinear form from the cartesian product of some functional space E to another space F. Then, as mentioned in [10], there are at least two natural ways of obtaining a quadratic representation Q on E. One can consider $Q(f) = q(f, f)$ or, for fixed ϕ, one can set $Q(f) = |q(f, \phi)|^2$.

The first of these methods is followed in order to pass from the "cross" Rihaczek sesquilinear form $R(f, g)(x, \omega) = e^{-2\pi i x\omega} f(x)\overline{\hat{g}(\omega)}$ to the (quadratic) Rihaczek representation (1.4), as well as from the "cross" Wigner distribution $Wig(f, g)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i x\omega} f(x + t/2)\overline{g(x - t/2)}dt$ to the quadratic Wigner representation (1.5).

The second is used to pass from the short-time Fourier transform $V_\phi f(x, \omega)$, which is sesquilinear in the couple (f, ϕ), to the spectrogram $Sp_\phi(f)(x, \omega) = |V_\phi f(x, \omega)|^2$.

We propose here a third way of defining a quadratic representation. Let q be a given sesquilinear form, then, for fixed ϕ and ψ, we set

$$Q_{\phi,\psi}(f)(x, \omega) = q(f, \phi)\overline{q(f, \psi)}. \tag{2.1}$$

We observe that, in the case $\phi = \psi$ our definition coincides obviously with the second method mentioned above. On the other hand it can also be viewed as an application of the first method to the sesquilinear form $\tilde{q}(f,g) = q(f,\phi)\overline{q(g,\psi)}$.

In order to justify our definition we shall now focus on the case where $q(f,\phi) = V_\phi f$ and introduce some considerations about the spreading effect of the spectrogram. As we mentioned in the previous section, the presence of the window ϕ is responsable for this effect. More precisely, as a consequence of the uncertainty principle, if the support of the window ϕ is well concentrated around the origin then we have a little spreading effect in time, i.e. a good "localization" in time, but a considerable spreading effect in the frequency, i.e. a bad "localization" in frequency. Vice versa, if the support of ϕ is "widespread" then the localization is bad in time but good in frequency. This suggests that instead of the spectrogram $Sp_\phi(f) = |V_\phi f|^2 = V_\phi f \overline{V_\phi f}$, we can fix a second window ψ and consider the form

$$Q_{\phi,\psi}(f) = V_\phi f \overline{V_\psi f}. \tag{2.2}$$

The advantage is that, if ϕ is very concentrated around the origin and ψ very spread on the plane (i.e. $\hat{\psi}$ is very concentrated), then $V_\phi f$ will show a good localization in time and $V_\psi f$ a good localization in frequency. Due to the reciprocal cut-off effect in the product, $V_\phi f \overline{V_\psi f}$ will have both good localizations. The drawback is of course the lost of positivity of the distribution. In the limit case where ϕ tends to δ and ψ to 1 then we shall have no spreading effect at all i.e. the third condition in (1.1) will be exactly satisfied. We shall prove that this case will coincide with that of the Rihaczek representation.

The better behaviour with regard to the support property of the representation $Q_{\phi,\psi}$ compared with the spectrogram can be made precise in many different ways. For example we can remark that $V_\phi f = (e^{-2\pi i\omega(\cdot)} f) * \overline{g}$ and therefore, from the support property of the convolution, $\Pi_x\,(\text{supp } V_\phi f) \subset \overline{\text{supp } \phi + \text{supp } f}$. From the fundamental equality of time-frequency analysis $V_\phi f(x,\omega) = e^{-2\pi i x\omega} V_{\hat{\phi}} \hat{f}(\omega, -x)$ we also have $\Pi_\omega\,(\text{supp } V_\phi f) \subset \overline{\text{supp } \hat{\phi} + \text{supp } \hat{f}}$, so we have proved the following property.

Proposition 2.1. *Let $B_0^{\delta_j}$ indicate balls in $\mathbb{R}^d$ of radius $\delta_j > 0$, $j = 1, 2$.*
i) If $\text{supp } \phi \subset B_0^{\delta_1}$ *then* $\Pi_x\,(\text{supp } Sp_\phi(f)) \subset \text{supp } f + B_0^{\delta_1}$;
ii) If $\text{supp } \hat{\phi} \subset B_0^{\delta_2}$ *then* $\Pi_\omega\,(\text{supp } Sp_\phi(f)) \subset \text{supp } \hat{f} + B_0^{\delta_2}$.

Of course hypothesis i) and ii) of this proposition can not be both satisfied, which is one possible form of expressing that for the spectrogram good localization in time and in frequency are complementary.

An immediate consequence of this same proposition is that, on the contrary, for the representation $Q_{\phi,\psi} = V_\phi f \overline{V_\psi f}$ we can have arbitrary good localization both in time and frequency, more precisely:

Proposition 2.2. *Let $B_0^{\delta_j}$ indicate balls in $\mathbb{R}^d$ of radius $\delta_j > 0$, $j = 1, 2$.*
Suppose that $\text{supp } \phi \subset B_0^{\delta_1}$ *and* $\text{supp } \hat{\psi} \subset B_0^{\delta_2}$, *then*

i) $\Pi_x\,(\text{supp } Q_{\phi,\psi}(f)) \subset \text{supp } f + B_0^{\delta_1}$;
ii) $\Pi_\omega\,(\text{supp } Q_{\phi,\psi}(f)) \subset \text{supp } \hat{f} + B_0^{\delta_2}$.

We present next two possible functional settings for the representation $Q_{\phi,\psi}$. We start by recalling some properties of the STFT. Let $f \otimes \phi$ be the (tensor) product $f \otimes \phi\,(x,t) = f(x)\phi(t)$, τ_a the asymmetric coordinate transform $\tau_a F(x,t) = F(t, t-x)$, and $\mathcal{F}_2$ the partial Fourier transform $\mathcal{F}_2 F(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i t\omega} F(x,t)\,dt$ of a function F on $\mathbb{R}^{2d}$. Then an easy computation gives (see [7]):

Lemma 2.3. *If $f, \phi \in \mathrm{L}^2(\mathbb{R}^d)$, then*

$$V_\phi f(x,\omega) = \mathcal{F}_2 \tau_a (f \otimes \overline{\phi}) \quad .$$

Lemma 2.3 is used to extend the STFT to a map from $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Namely, if $f, \phi \in \mathcal{S}'(\mathbb{R}^d)$, then $f \otimes \overline{\phi} \in \mathcal{S}'(\mathbb{R}^{2d})$, with $(f \otimes \overline{\phi}, \chi) = (f_x, \overline{(\overline{\phi}_\omega, \chi)})$, $\chi \in \mathcal{S}(\mathbb{R}^{2d})$, as we consider distributions as anti-linear functionals. Further both operators τ_a and $\mathcal{F}_2$ are isomorphisms on $\mathcal{S}'(\mathbb{R}^{2d})$. Thus $V_\phi f$ is a well-defined tempered distribution, whenever $f, \phi \in \mathcal{S}'(\mathbb{R}^d)$.
Following these general lines $Q_{\phi,\psi}(f,g) = V_\phi f \overline{V_\psi g}$ would result as a product of two temperated distributions. To overcome problems of definition we introduce then some restrictions.

Definiton 2.4. *Let $\mathcal{B}^\infty(\mathbb{R}^d)$ be the space of smooth bounded functions together with all its derivatives, i.e. the space of C^∞ functions h such that for every multi-index $\alpha = (\alpha_1, \ldots \alpha_d) \in \mathbb{N}^d$ there exists a constant M_α with $|\partial_x^\alpha h(x)| \le M_\alpha$ on $\mathbb{R}^d$.*

By the differentiation under the integral and the Riemann-Lebesgue theorem one immediately obtain the following result.

Lemma 2.5. *If $\psi \in \mathcal{B}^\infty(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, then $V_\psi g \in \mathcal{B}^\infty(\mathbb{R}^{2d})$.*

Of course the role of g and ψ in Lemma 2.5 can be exchanged .

We observe now that the form $q_{\phi,\psi}(f,g) = V_\phi f \overline{V_\psi g}$ makes sense whenever $V_\phi f$ is a tempered distribution and $V_\psi g \in \mathcal{B}^\infty(\mathbb{R}^d)$. From Lemma 2.5 this is the case for example of the next proposition.

Proposition 2.6. *Let $f, \phi \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, $\psi \in \mathcal{B}^\infty(\mathbb{R}^d)$ (or vice versa $g \in \mathcal{B}^\infty(\mathbb{R}^d)$, $\psi \in \mathcal{S}(\mathbb{R}^d)$). Then $q_{\phi,\psi}(f,g)$ is a well defined tempered distribution in $\mathcal{S}'(\mathbb{R}^{2d})$.*

Better regularity for the time-frequency representation $q_{\phi,\psi}$ can easily be obtained for example in the following situation.

Proposition 2.7. *Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $\phi, \psi \in \mathcal{S}'(\mathbb{R}^d)$ (or vice versa $f, g \in \mathcal{S}'(\mathbb{R}^d)$, and $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$), then $q_{\phi,\psi}(f,g) \in C^\infty(\mathbb{R}^{2d})$.*

As a second functional setting for the representation (2.2) we consider L^p spaces.
We shall need the following boundedness result about the STFT, see [14], [1].

Proposition 2.8. *Let us fix q and p satisfying $q \geq 2$ and $q' \leq p \leq q$, where q' is the conjugate of q, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$. Then $V : L^{p'}(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^{2d})$ is continuous, in particular*

$$\|V_\phi f\|_{L^q} \leq \|f\|_{L^{p'}} \|\phi\|_{L^p}. \tag{2.3}$$

We can now expresses the behaviour of (2.2) in the context of L^p spaces by the following proposition.

Theorem 2.9. *Let $q \in [1, \infty]$, $q_j \in [2, \infty]$, $p_j \in [1, \infty]$, $j = 1, 2$ satisfy $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, and $q'_j \leq p_j \leq q_j$, where q'_j is the conjugate of q_j, i.e. $\frac{1}{q_j} + \frac{1}{q'_j} = 1$. Then*

$$q : (f, \phi, g, \psi) \in L^{p'_1}(\mathbb{R}^d) \times L^{p_1}(\mathbb{R}^d) \times L^{p'_2}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d) \to q_{\phi,\psi}(f, g) \in L^q(\mathbb{R}^{2d}) \tag{2.4}$$

is continuous, in particular

$$\|q_{\phi,\psi}(f, g)\|_q \leq \|f\|_{L^{p'_1}} \|\phi\|_{L^{p_1}} \|g\|_{L^{p'_2}} \|\psi\|_{L^{p_2}} \tag{2.5}$$

Proof. We begin by remarking that $q = \frac{q_1 q_2}{q_1 + q_2} \geq 1$ for $q_j \geq 2$, $j = 1, 2$, which means that (2.4) makes sense under our hypothesis. From Proposition 2.8 we have that $V_\varphi h : L^{p'}(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^{2d})$, with $q \geq 2$ and $q' \leq p \leq q$, is continuous, and in particular $\|V_\varphi h\|_{L^q} \leq \|h\|_{L^{p'}} \|\varphi\|_{L^p}$ holds. So we obtain that

$$q = V_\phi f \overline{V_\psi g} : \quad L^{p'_1}(\mathbb{R}^d) \times L^{p_1}(\mathbb{R}^d) \times L^{p'_2}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d) \quad \to \quad L^{q_1}(\mathbb{R}^{2d}) \cdot L^{q_2}(\mathbb{R}^{2d})$$

with

$$\|V_\phi f\|_{L^{q_1}} \leq \|f\|_{L^{p'_1}} \|\phi\|_{L^{p_1}}, \quad \|\overline{V_\psi g}\|_{L^{q_2}} \leq \|g\|_{L^{p'_2}} \|\psi\|_{L^{p_2}} .$$

where $L^{q_1}(\mathbb{R}^{2d}) \cdot L^{q_2}(\mathbb{R}^{2d})$ indicates the subset of $\mathcal{S}'(\mathbb{R}^d)$ of all the products fg of a function $f \in L^{q_1}(\mathbb{R}^{2d})$ with a function $g \in L^{q_2}(\mathbb{R}^{2d})$.

The generalized Hölder's formula:

$$\|V_\phi f \overline{V_\psi g}\|_q \leq \|V_\phi f\|_{L^{q_1}} \|\overline{V_\psi g}\|_{L^{q_2}}, \quad \text{for } \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \tag{2.6}$$

proves then (2.5). Moreover (2.6) means:

$$L^{q_1}(\mathbb{R}^{2d}) \cdot L^{q_2}(\mathbb{R}^{2d}) \hookrightarrow L^q(\mathbb{R}^{2d}) ,$$

which proves (2.4). □

We end this section by showing that the form (2.2) represents a link between the spectrogram $Sp_\phi(f)$ in (1.3) and the Rihaczek's distribution $R(f)$ in (1.4).

Theorem 2.10. *Let $f \in \mathcal{S}(\mathbb{R}^d)$, and $\phi \in \mathcal{S}'(\mathbb{R}^d)$, then*
i) $Q_{\phi,\phi}(f) = Sp_\phi(f)$;
ii) $Q_{\delta,1}(f) = R(f)$.

Proof. The choice $\phi = \psi$ in (2.2) proves immediately the first part of the theorem. Consider now the limit cases $\phi = \delta$, the point measure $(\delta, \varphi) = \overline{\varphi(0)}$, and $\psi = 1$. For $\chi \in \mathcal{S}(\mathbb{R}^{2d})$ we have:

$$\begin{aligned}(V_\delta f, \chi) &= (\mathcal{F}_2 \tau_a (f \otimes \overline{\delta}), \chi) = (f \otimes \overline{\delta}, {\tau_a}^{-1} {\mathcal{F}_2}^{-1} \chi) = (f, \overline{(\overline{\delta}, {\tau_a}^{-1} {\mathcal{F}_2}^{-1} \chi)}) \\ &= (f, \textstyle\int_{\mathbb{R}^d} e^{2\pi i x\omega} \chi(x - t, \omega)\, d\omega \,|_{t=0}) = (f, \int_{\mathbb{R}^d} e^{2\pi i x\omega} \chi(x, \omega)\, d\omega) \\ &= \textstyle\int_{\mathbb{R}^{2d}} e^{-2\pi i x\omega} f(x) \overline{\chi(x, \omega)}\, dx d\omega = (e^{-2\pi i x\omega} f(x), \chi(x, \omega)) \ .\end{aligned}$$

Therefore

$$V_\delta f(x, \omega) = e^{-2\pi i x\omega} f(x). \tag{2.7}$$

Now by relation (2.7) we obtain

$$V_1 g(x, \omega) = e^{-2\pi i x\omega} V_{\hat{1}} \hat{g}(\omega, -x) = e^{-2\pi i x\omega} V_\delta \hat{g}(\omega, -x) = \hat{g}(\omega).$$

We conclude that $V_\delta f \overline{V_1 g}(x, \omega) = e^{-2\pi i x\omega} f(x) \overline{\hat{g}(\omega)}$. □

We observe that it is of course possible to define explicitly a "path" between the spectrogram and the Rihaczek representation. For example, using Gaussian functions, we can consider the L^1-normalized window $\phi_\lambda(x) = \lambda^{d/2} e^{-\pi\lambda x^2}$ and set $\psi_\lambda(x) = \hat{\phi}_\lambda(x) = e^{-\pi \frac{1}{\lambda} x^2}$ with $\lambda \in [1, \infty]$, adopting the convention $\phi_\infty = \delta$, $\psi_\infty = 1$.

3. Uncertainty Principles

In this section we examine various forms of the uncertainty principle. We study in particular which forms of this principle apply to the quadratic representation $Q_{\phi,\psi}$ and show that suitable L^p formulations have, as limit case, uncertainty principles for the spectrogram and for the Rihaczek representation. We begin with recalling the classical uncertainty principle.

Proposition 3.1. *For $f \in L^2(\mathbb{R})$, $a, b \in \mathbb{R}$ we have*

$$\left(\int_{\mathbb{R}} (x - a)^2 |f(x)|^2 \ dx \right)^{1/2} \left(\int_{\mathbb{R}} (\omega - b)^2 |\hat{f}(\omega)|^2 \ d\omega \right)^{1/2} \geq \frac{1}{4\pi} \|f\|_{L^2}^2$$

As pointed out in [11] this principle is essentially an inequality for the Rihaczek representation and can actually be reformulated as:

$$\left(\int_{\mathbb{R}^2} (x - a)^2 (\omega - b)^2 |Rf(x, \omega)|^2 \ dx d\omega \right)^{1/2} \geq \frac{1}{4\pi} \|f\|_{L^2}^2$$

Another expression of the uncertainty principle for the Rihaczek representation is the well-known uncertainty principle of Donoho-Stark for which we refer to [6].

We prove next a form of the uncertainty principle for the Rihaczek representation in terms of the integral $\int_U |R(f)(z)|\ dz$ instead of $\int_U |R(f)(z)|^2\ dz$ (as usual, $z = (x, \omega) \in \mathbb{R}^{2d}$).

Proposition 3.2. *Let $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\epsilon \geq 0$. If $U \subset \mathbb{R}^{2d}$ is a measurable set and*

$$(1-\epsilon)\|f\|_{L^1}\|f\|_{L^\infty} \leq \int_U |Rf(z)|dz,$$

then $\mu(U) \geq 1 - \epsilon$, where $\mu(U)$ is the Lebesgue measure of U.

Proof. As $|Rf(z)| = |f(x)\hat{f}(\omega)|$ we have immediately:

$$(1-\epsilon)\|f\|_{L^\infty}\|f\|_{L^1} \leq \int_U |Rf(z)|dz \leq \|f\|_{L^\infty}\|\hat{f}\|_{L^\infty}\int_U\ dz \leq \|f\|_{L^\infty}\|f\|_{L^1}\ \mu(U)$$

□

For $p \in (1, \infty)$ we have the following result.

Proposition 3.3. *Let $f \in L^p(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)$, U be measurable with finite measure and $\epsilon \geq 0$. If*

$$(1-\epsilon)\|f\|_{L^p}\|f\|_{L^{p'}} \leq \int_U |Rf(z)|dz,$$

then $\mu(U) \geq C_{p_0}^{dp_0}(1-\epsilon)^{p_0}$, where $p_0 = \min(p, p')$ and $C_{p_0} = \sqrt{\frac{(p_0')^{1/p_0'}}{(p_0)^{1/p_0}}}$.

Proof. In the proof we make use of the continuous inclusion $L^{p_2}(U_j) \subset L^{p_1}(U_j)$ for $p_1 \leq p_2$, where $\|f\|_{L^{p_2}} \leq \mu(U_j)^{(1/p_1 - 1/p_2)}\|f\|_{L^{p_1}}$ and, of the boundedness of the Fourier transform $\mathcal{F} : L^q(\mathbb{R}^d) \to L^{q'}(\mathbb{R}^d)$ with $\|\hat{f}\|_{L^{q'}} \leq \left(\sqrt{q^{1/q}/q'^{1/q'}}\right)^d \|f\|_{L^q}$, which holds for $q \in [1, 2]$.
We consider at first the case $p \leq 2$, we have the estimates
(3.1)

$$\begin{aligned}(1-\epsilon)\|f\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)} &\leq \textstyle\int_U |f(x)\hat{f}(\omega)|\ dxd\omega \\ &\leq \|f \otimes \hat{f}\|_{L^{p'}(U)}\mu(U)^\alpha \\ &\leq \|f\|_{L^{p'}(\mathbb{R}^d)}\|\hat{f}\|_{L^{p'}(\mathbb{R}^d)}\mu(U)^\alpha \\ &\leq \left(\sqrt{p^{1/p}/p'^{1/p'}}\right)^d \|f\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)}\mu(U)^\alpha\end{aligned}$$

where $\alpha = 1 - 1/p' = 1/p$, which proves the thesis.

Analogously for $p \geq 2$ we have

(3.2)

$$\begin{aligned}(1-\epsilon)\|f\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)} &\leq \textstyle\int_U |f(x)\hat{f}(\omega)|\ dxd\omega \\ &\leq \|f \otimes \hat{f}\|_{L^p(U)}\mu(U)^\beta \\ &\leq \|f\|_{L^p(\mathbb{R}^d)}\|\hat{f}\|_{L^p(\mathbb{R}^d)}\mu(U)^\beta \\ &\leq \left(\sqrt{p'^{1/p'}/p^{1/p}}\right)^d \|f\|_{L^p(\mathbb{R}^d)}\|f\|_{L^{p'}(\mathbb{R}^d)}\mu(U)^\beta\end{aligned}$$

where $\beta = 1 - 1/p = 1/p'$. The thesis follows then from (3.1) and (3.2). □

Remark 3.4. *We note that we can regard Proposition 3.2 as a particular case of Proposition 3.3 if we allow* $p \in [1,\infty]$*, adopting, in the case* $p_0 = 1$*, the convention* $\infty^{1/\infty} = 0^{1/0} = 1$.

Our next goal is the investigation of possible relations between these results and corresponding uncertainty principles for our general class of representations $Q_{\phi,\psi}$. For this class we prove now the following simple uncertainty principle.

Proposition 3.5. *Consider non zero functions* $\phi \in L^p(\mathbb{R}^d)$*,* $\psi \in L^{p'}(\mathbb{R}^d)$ *and* $f \in L^p(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)$*, where* $1 \le p \le \infty$*. Let* U *be a measurable set in* $\mathbb{R}^{2d}$ *and* $\epsilon \ge 0$*. If*

$$(1-\epsilon)\|f\|_{L^p}\|f\|_{L^{p'}}\|\phi\|_{L^p}\|\psi\|_{L^{p'}} \le \int_U |Q_{\psi,\phi}(f)(z)|\, dz, \tag{3.3}$$

then $\mu(U) \ge 1-\epsilon$.

Proof. Writing, as usual, $z = (x,\omega)$, $\phi_z(t) = e^{2\pi i t\omega}\phi(t-x)$, $\psi_z(t) = e^{2\pi i t\omega}\psi(t-x)$, we have

$$\begin{aligned}(1-\epsilon)\|f\|_{L^p}\|f\|_{L^{p'}}\|\phi\|_{L^p}\|\psi\|_{L^{p'}} \le {} & \textstyle\int_U |Q_{\psi,\phi}(f)(z)|\, dz = \\ \textstyle\int_U |(f,\phi_z)(\psi_z,f)|\, dz \le {} & \|f\|_{L^p}\|f\|_{L^{p'}}\|\phi\|_{L^p}\|\psi\|_{L^{p'}} \textstyle\int_U dz = \\ & \|f\|_{L^p}\|f\|_{L^{p'}}\|\phi\|_{L^p}\|\psi\|_{L^{p'}}\mu(U).\end{aligned} \tag{3.4}$$

□

Though its simplicity, this result is enlightening for what concerns the connections with the uncertainty principles for the spectrogram and the Rihaczek representation. Actually, for $\psi = \phi \in L^2(\mathbb{R}^d)$, it contains the well-known weak-uncertainty principle for the spectrogram (see [14]), namely:

Proposition 3.6. *For* $f, \phi \in L^2(\mathbb{R}^d)$*,* U *measurable set in* $\mathbb{R}^{2d}$*,* $\epsilon \ge 0$*, the condition*

$$(1-\epsilon)\|f\|_{L^2}^2\|\phi\|_{L^2}^2 \le \int_U |V_\phi f(z)|^2\, dz \tag{3.5}$$

implies $\mu(U) \ge 1-\epsilon$.

On the other hand, for $p = 1$, keeping $\|\phi\|_{L^1} = \|\psi\|_{L^\infty} = 1$, we can let ϕ tend to δ and ψ tend to 1 in $\mathcal{S}'(\mathbb{R}^d)$ and we see therefore from Theorem 2.10 that the previous proposition yields, in this limit case, exactly the uncertainty principle of Proposition 3.2 for the Rihaczek representation.

Proposition 3.5 also permits to understand the substantial reason why, in the cases $1 < p < \infty$, we do not recapture the uncertainty principle of the Rihaczek representation as a limit case of that of the generalized spectrogram. Namely, if $\psi \in L^{p'}(\mathbb{R}^d)$ tends to 1 in $\mathcal{S}'(\mathbb{R}^d)$, then $\|\psi\|_{L^{p'}}$ can not remain bounded, otherwise there would exist a subsequence weakly convergent to an element $\psi_0 \in L^{p'}(\mathbb{R}^d)$, contradicting $\psi \to 1$ in $\mathcal{S}'(\mathbb{R}^d)$.

In order that the product $\|\phi\|_{L^p}\|\psi\|_{L^{p'}}$, which appears in (3.3), remains bounded, it is then necessary that $\|\phi\|_{L^p} \to 0$. This implies $\phi \to 0$ in $\mathcal{S}'(\mathbb{R}^d)$, and is therefore incompatible with the condition $\phi \to \delta$. This justifies the more involved result for the Rihaczek representation which we have obtained in Proposition 3.3.

4. Marginal Distributions

In this section we consider the quadratic form

$$Q_{\phi,\psi}(f,f) = V_\phi f \overline{V_\psi f},$$

and we want to study its marginal distributions, i.e.

$$Q^{(1)}_{\phi,\psi}(x) = \int V_\phi f \overline{V_\psi f}\, d\omega, \tag{4.1}$$

and

$$Q^{(2)}_{\phi,\psi}(\omega) = \int V_\phi f \overline{V_\psi f}\, dx. \tag{4.2}$$

The marginals (4.1) and (4.2) are well defined for instance if $V_\phi f \overline{V_\psi f} \in L^1(\mathbb{R}^{2d})$, that is satisfied when q_1, q_2 in Theorem 2.9 verify $\frac{q_1 q_2}{q_1+q_2} = 1$, that means $q_1 = q_2'$. Since $q_j \geq 2$, $j = 1, 2$ we have to choose $q_1 = q_2 = 2$, and consequently $p_1 = p_2 = 2$. Then from now on we shall fix $f, \phi, \psi \in L^2(\mathbb{R}^d)$.

We start now by giving the explicit expression of (4.1) and (4.2) and then we study their average E and their standard deviation σ^2.

Proposition 4.1. *For every $f \in L^2(\mathbb{R}^d)$ and $\phi, \psi \in L^2(\mathbb{R}^d)$ we have:*

$$\int V_\phi f \overline{V_\psi f}\, d\omega = |f|^2 * \big(\tilde{\psi}\,\overline{\tilde{\phi}}\big)(x), \tag{4.3}$$

where as usual $\tilde{\phi}(x) = \phi(-x)$ and $\tilde{\psi}(x) = \psi(-x)$.

Proof. We start by considering $f, \phi, \psi \in \mathcal{S}(\mathbb{R}^d)$, recovering then (4.3) in the generical case by density arguments. By a simple change of variables we get:

$$\int V_\phi f \overline{V_\psi f}\, d\omega = \int e^{-2\pi i q\omega} f(q+s)\,\overline{f(s)}\,\psi(s-x)\,\overline{\phi(q+s-x)}\, ds\, dq\, d\omega;$$

now, by integrating at first in the ω-variable, since $\int e^{-2\pi i q\omega}\, d\omega$ is the δ distribution in the q-variable, we obtain that

$$\int V_\phi f \overline{V_\psi f}\, d\omega = \int f(s)\,\overline{f(s)}\,\psi(s-x)\,\overline{\phi(s-x)}\, ds = |f|^2 * \big(\tilde{\psi}\,\overline{\tilde{\phi}}\big)(x);$$

that is what we want to prove. □

Let us analyze now (4.2). From the previous Proposition and the formula $V_g f(x,\omega) = e^{-2\pi i\omega x} V_{\hat{f}}\hat{g}(\omega, -x)$ we get immediately an analogous result.

Proposition 4.2. *For every* $f \in L^2(\mathbb{R}^d)$ *and* $\phi, \psi \in L^2(\mathbb{R}^d)$ *we have:*

$$\int V_\phi f \overline{V_\psi f}\, dx = |\hat{f}|^2 * \big(\hat{\tilde{\psi}}\, \overline{\hat{\tilde{\phi}}}\big)(\omega), \tag{4.4}$$

with the same notations as in the previous proposition.

In formula (4.3) we obtain the marginal with respect the x-variable, $|f(x)|^2$, if $\big(\tilde{\psi}\,\overline{\tilde{\phi}}\big)(x) = \delta$, $(\delta, \varphi) = \overline{\varphi(0)}$, which is the case when

$$\phi = \delta \text{ and } \psi(0) = 1 \text{ or vice versa } \psi = \delta \text{ and } \phi(0) = 1\ .$$

In formula (4.4) we obtain the marginal with respect the ω-variable, $|\hat{f}(\omega)|^2$, if $\big(\hat{\tilde{\psi}}\,\overline{\hat{\tilde{\phi}}}\big)(\omega) = \delta$, which happens when

$$\hat{\phi} = \delta \text{ and } \hat{\psi}(0) = 1 \text{ or vice versa } \hat{\psi} = \delta \text{ and } \hat{\phi}(0) = 1\ .$$

We have therefore:

Corollary 4.3. *For the generalized spectrogram* $Q_{\phi,\psi}(f)$ *both marginal distributions conditions are satisfied in the cases* $\phi = \delta$, $\psi = 1$ *and* $\phi = 1$, $\psi = \delta$, *which correspond to the Rihaczek's distribution* $R(f) = e^{-2\pi i x\omega} f(x)\overline{\hat{f}(\omega)}$ *and its conjugate* $\overline{R(f)}$ *respectively. More generally, for every multi-index* α, *the marginal distributions conditions are satisfied by generalized spectrograms with windows* $\phi = \partial^\alpha \delta$ *and* $\psi = x^\alpha/\alpha!$, *or viceversa* $\phi = x^\alpha/\alpha!$ *and* $\psi = \partial^\alpha \delta$.

We analyze now some properties of the marginal distributions (4.1) and (4.2).

Proposition 4.4. *We have:*

$$E\big(Q^{(1)}_{\phi,\psi}\big) = E\big(|f|^2\big)\big(\psi, \phi\big)_{L^2} - \|f\|^2_{L^2}\big(x\psi, \phi\big)_{L^2} \tag{4.5}$$

and

$$\begin{aligned} \sigma^2\big(Q^{(1)}_{\phi,\psi}\big) = &\int \Big[y - E\big(Q^{(1)}_{\phi,\psi}\big)\Big]^2 |f(y)|^2\, dy \quad \big(\psi, \phi\big)_{L^2} \\ &- 2\int \Big[y - E\big(Q^{(1)}_{\phi,\psi}\big)\Big] |f(y)|^2\, dy \quad \big(x\psi, \phi\big)_{L^2} \\ &+ \|f\|^2_{L^2}\ \big(x^2\psi, \phi\big)_{L^2} \end{aligned} \tag{4.6}$$

Proof. By (4.3) and a simple exchange of integration order we have:

$$\begin{aligned} E\big(Q^{(1)}_{\phi,\psi}\big) &= \int x\big(|f|^2 * (\tilde{\psi}\,\overline{\tilde{\phi}})(x)\big)\, dx \\ &= \int y|f(y)|^2\, \tilde{\psi}(x-y)\, \overline{\tilde{\phi}}(x-y) \ +\ (x-y)|f(y)|^2\, \tilde{\psi}(x-y)\, \overline{\tilde{\phi}}(x-y)\, dy\, dx \\ &= \big(\psi, \phi\big)_{L^2} \int y|f(y)|^2\, dy - \big(x\psi, \phi\big)_{L^2} \int |f(y)|^2\, dy, \end{aligned}$$

and so (4.5) is proved. Regarding (4.6) we observe that

$$\begin{aligned}
\sigma^2\big(Q^{(1)}_{\phi,\psi}\big) &= \int \Big[x - E\big(Q^{(1)}_{\phi,\psi}\big)\Big]^2 \big(|f|^2 * (\tilde{\psi}\,\overline{\tilde{\phi}})(x)\big)\,dx \\
&= \int \Big[y - E\big(Q^{(1)}_{\phi,\psi}\big) + x - y\Big]^2 |f(y)|^2\,\tilde{\psi}(x-y)\,\overline{\tilde{\phi}}(x-y)\,dy\,dx \\
&= \int \Big[y - E\big(Q^{(1)}_{\phi,\psi}\big)\Big]^2 |f(y)|^2\,\tilde{\psi}(x-y)\,\overline{\tilde{\phi}}(x-y)\,dy\,dx \\
&\quad + 2\int (x-y)\Big[y - E\big(Q^{(1)}_{\phi,\psi}\big)\Big] |f(y)|^2\,\tilde{\psi}(x-y)\,\overline{\tilde{\phi}}(x-y)\,dy\,dx \\
&\quad + \int (x-y)^2 |f(y)|^2\,\tilde{\psi}(x-y)\,\overline{\tilde{\phi}}(x-y)\,dy\,dx \\
&= \int \Big[y - E\big(Q^{(1)}_{\phi,\psi}\big)\Big]^2 |f(y)|^2\,dy\ \big(\psi,\phi\big)_{L^2} \\
&\quad - 2\int \Big[y - E\big(Q^{(1)}_{\phi,\psi}\big)\Big] |f(y)|^2\,dy\ \big(x\psi,\phi\big)_{L^2} \\
&\quad + \int |f(y)|^2\,dy\ \big(x^2\psi,\phi\big)_{L^2},
\end{aligned}$$

that is what we wanted to prove. □

With analogous computations results as in Proposition 4.4 can be proved also for $Q^{(2)}_{\phi,\psi}(\omega)$, namely:

Proposition 4.5. *We have:*

$$E\big(Q^{(2)}_{\phi,\psi}\big) = E\big(|\hat{f}|^2\big)\big(\psi,\phi\big)_{L^2} - \|f\|^2_{L^2}\big(\omega\hat{\psi},\hat{\phi}\big)_{L^2} \tag{4.7}$$

and

$$\begin{aligned}
\sigma^2\big(Q^{(2)}_{\phi,\psi}\big) = &\int \Big[\eta - E\big(Q^{(2)}_{\phi,\psi}\big)\Big]^2 |\hat{f}(\eta)|^2\,d\eta\ \big(\psi,\phi\big)_{L^2} \\
&- 2\int \Big[\eta - E\big(Q^{(2)}_{\phi,\psi}\big)\Big] |\hat{f}(\eta)|^2\,d\eta\ \big(\omega\hat{\psi},\hat{\phi}\big)_{L^2} \\
&+ \|f\|^2_{L^2}\big(\omega^2\hat{\psi},\hat{\phi}\big)_{L^2}.
\end{aligned} \tag{4.8}$$

Remark 4.6. *Let us remark that in the particular case when*

$$\big(\psi,\phi\big)_{L^2} = 1, \quad \big(x\psi,\phi\big)_{L^2} = 0, \tag{4.9}$$

that is verified for example when $\tilde{\psi}(x)\overline{\tilde{\phi}}(x)$ *is a Gaussian, we have from* (4.5) *that*

$$E\big(Q^{(1)}_{\phi,\psi}\big) = E\big(|f|^2\big); \tag{4.10}$$

analogously, when

$$\big(\psi,\phi\big)_{L^2} = 1, \quad \big(\omega\hat{\psi},\hat{\phi}\big)_{L^2} = 0 \tag{4.11}$$

we obtain from (4.7)

$$E\big(Q^{(2)}_{\phi,\psi}\big) = E\big(|\hat{f}|^2\big). \tag{4.12}$$

Moreover, if (4.9) *and*

$$\left(x^2\psi,\phi\right)_{L^2}=0 \tag{4.13}$$

are satisfied, we also have from (4.6) *and* (4.10) *that*

$$\sigma^2\big(Q^{(1)}_{\phi,\psi}\big)=\sigma^2\big(|f|^2\big),$$

and when the windows ϕ *and* ψ *satisfy* (4.11) *and*

$$\left(\omega^2\hat\psi,\hat\phi\right)_{L^2}=0 \tag{4.14}$$

then we get from (4.8) *and* (4.12)

$$\sigma^2\big(Q^{(2)}_{\phi,\psi}\big)=\sigma^2\big(|\hat f|^2\big)\ .$$

A simple example in the one dimensional case of a function $\tilde\psi(x)\overline{\tilde\phi}(x)=h(x)$ *satisfying the conditions* (4.9) *and* (4.13) *is given by* $h(x)=k(|x|)$*, where*

$$k(x)=\begin{cases}\frac{3}{4x^2} & x\in(\frac12,1)\\ -\frac{3}{4x^2} & x\in(1,\frac32)\\ 0 & elsewhere\ .\end{cases}$$

Acknowledgements. The authors are thankful to the referee for his valuable remarks and suggestions.

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