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ANALYTIC AND GEVREY SOLUTIONS OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

We prove some results of existence and regularity in analitic-Gevrey spaces for solutions of linear and non-linear partial differential equations. In particular we consider Navier-Stokes equations and perturbations of powers of the Schrödinger operator.

1. Introduction

An elementary, but instructive, example of the use of the Gevrey classes for the study of the non-linear equations of the Applied Mathematics is given by the Cauchy problem:

$$\partial_t^2 u = F(t, x, u, \partial_t u, \partial_{x_1} u, ..., \partial_{x_n} u),$$
$$u(0, x) = u_0(x), \ (\partial_t u)(0, x) = u_1(x), \tag{1.1}$$

where $x = (x_1, ..., x_n)$ are the space variables in a domain of $\mathbb{R}^n$. Assume for simplicity that the non-linear function F is analytic, but allow

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possibly that $d_x u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ is replaced in (1.1) by $Ad_x u$, where A is a matrix of classical pseudo-differential operators of order 0. Problems of the type (1.1) appear often in the applications, related to equations of fluids, or with different Physical meanings, according to the choice of F. It is well known that (1.1) is well-posed in the classical sense, i.e., for initial data in Sobolev spaces or other standard function or distribution spaces, only if restrictive assumptions are satisfied by the right-hand side F. In other relevant cases, despite the solution is expected by Physical intuition, a theorem of existence is not possible in this frame.

Gevrey classes provide a different functional frame, allowing to satisfy the natural hope, that (1.1) should be always solvable, independently of F. Namely, if the initial data $u_0(x), u_1(x)$ belong to the Gevrey class G^s, with index $1 \leq s \leq 2$, than (1.1) admits a solution $u(t, x), t \leq T$, in the same Gevrey class. Note that, if $u_0(x), u_1(x)$ belong to G^1, i.e., are analytic functions, the existence of an analytic solution $u(t, x)$ is already granted by the Cauchy-Kowalevsky theorem.

To be more definite, let us recall here the definition of the Gevrey classes, which play the role of intermediate spaces between the spaces of the C^∞ and analytic functions. Given Ω, open subset of $\mathbb{R}^n$, we say that $u \in G^s(\Omega), s \geq 1$, if $u \in C^\infty(\Omega)$ and for every compact subset K of Ω we have

$$\sup_{x \in K} |\partial^\alpha u(x)| \leq C^{|\alpha|+1}(\alpha!)^s, \tag{1.2}$$

for a constant C depending only on u and K. When $s = 1$ we recapture the analytic case, whereas for $s > 1$ we obtain larger spaces, containing functions with compact support. It is interesting to observe that if $u \in G_0^s(\mathbb{R}^n)$, i.e., $u \in G^s(\mathbb{R}^n)$ has compact support, then its Fourier transform $\hat{u}(\xi)$ satisfies the estimates

$$|\hat{u}(\xi)| \leq Ce^{-\varepsilon|\xi|^{1/s}}, \ \xi \in \mathbb{R}^n, \tag{1.3}$$

for some positive constants C and ε. Actually, the estimates (1.3) characterize the G^s-regularity of a Fourier transformable function, or distribution. Observe that (1.3) implies

$$\|e^{\tau|\xi|^{1/s}}\hat{u}(\xi)\|^2 = \int e^{2\tau|\xi|^{1/s}}|\hat{u}(\xi)|^2 d\xi < \infty \tag{1.4}$$

for a sufficiently small $\tau > 0$. In the sequel we shall also consider Gevrey classes on the n-torus $\mathbb{T}^n$. Given a function on $\mathbb{T}^n$

$$u(x) = \sum_{\alpha \in \mathbb{Z}^n} u_\alpha e^{i\alpha x}, \ \ u_\alpha \in \mathbb{C}, \tag{1.5}$$

we have $u \in G^s(\mathbb{T}^n)$, $s \geq 1$, if and only if

$$\sum_{\alpha \in \mathbb{Z}^n} e^{2\tau|\alpha|^{1/s}}|u_\alpha|^2 < \infty \tag{1.6}$$

for some $\tau > 0$. The problem (1.1) can be obviously reset for $x \in \mathbb{T}^n$, with identical results of existence in $G^s(\mathbb{T}^n)$, $1 \leq s \leq 2$. For all the related proofs we address the reader to classical works, see Kajitani [10] and the references there; for the linear case see also the monographies of Rodino [19], Mascarello and Rodino [14]. In the sequel of the paper we want to show the effectiveness of the Gevrey functional frame in the study of some other linear and non-linear partial differential equations, with emphasis on equations of the Mathematical Physics. Namely, in the next Section 2 we review a result of Foias and Temam [6] concerning Navier-Stokes equations. In that paper existence of solutions is assumed already achieved in a certain Sobolev space by other methods, let us refer to Solonnikov [20, 21], and the aim is to show the Gevrey regularity of such solutions.

The other Sections of the present article are devoted to prove new results for perturbations of powers of the Schrödinger operator (without potential), in one space variable:

$$S = i\partial_t - \partial_x^2. \tag{1.7}$$

In fact, the local properties of

$$Su = F(t, x, u, \partial_x u) \tag{1.8}$$

do not differ from those of the non-perturbed operator S, see for example Messina and Rodino [15]. On the contrary, for the equations

$$S^2 u = F(t,\ x,\ \partial_t^l \partial_x^j u)_{2l+j \le 3} \tag{1.9}$$

we have completely different phenomena, depending on the lower order terms in the (linear or non-linear) expression of F. So for example we prove that

$$S^2 u = i \partial_t \partial_x u \tag{1.10}$$

is hypoelliptic, i.e., all the solutions are smooth, whereas the solutions of $S^2 u = 0$ admit obviously singularities.

Gevrey classes provide a frame for the unified treatment of (1.9): leaving a more detailed analysis to the future, we begin to give in the last part of this paper some simple but representative results of Gevrey solvability. The proofs rely on the techniques introduced in De Donno and Oliaro [4].

2. Navier-Stokes Equations

In the following we report the results of Foias and Temam [6] in a lightly more general form, as suggested by the final Section of that paper. We do not give proofs, since they are essentially a repetition of those in [6].

We consider the Navier-Stokes equations of viscous incompressible fluids with space periodicity boundary conditions, in space dimension $n = 2$ or $n = 3$. Setting then the problem in $[0,\ T] \times \mathbb{T}^n$ we obtain for the unknown function $u = (u^{(1)},\ u^{(2)})$ for $n = 2$, or $u = (u^{(1)},\ u^{(2)},\ u^{(3)})$ for $n = 3$, the abstract evolution equation

$$\frac{du}{dt} + \nu A u + B(u) = f, \ \ u(0) = u_0, \tag{2.1}$$

where ν is the kinematic viscosity and $A,\ B$ are defined as standard. Namely, we refer to the Hilbert space H defined as the closed subspace of $L^2(\mathbb{T}^n)^n$ of all functions

$$u = \sum_{\alpha \in \mathbb{Z}^n} u_\alpha e^{i\alpha x},\ u_\alpha \in \mathbb{C}^n,\ u_{-\alpha} = \overline{u}_\alpha,\ u_0 = 0, \tag{2.2}$$

such that for every $\alpha \in \mathbb{Z}^n$

$$\alpha \cdot u_\alpha = \sum_{j=1}^{n} \alpha_j u_\alpha^{(j)} = 0 \tag{2.3}$$

(i.e., $\nabla u = 0$). The operator A in (2.1) is the Stokes operator with space periodicity boundary conditions. It is a self-adjoint unbounded positive operator in H. The domain $D(A) \subset H$ and more generally the domain of the positive power $D(A^{r/2})$, $r > 0$, is the set of all $u \in H$ such that

$$\sum_{\alpha \in \mathbb{Z}^n} |\alpha|^{2r} |u_\alpha|^2 < \infty. \tag{2.4}$$

We then consider $(G^s(\mathbb{T}^n))^n$, the space of all the n-tuple of functions satisfying (1.6). More generally, given $s \geq 1$, $\tau > 0$ and $r \geq 0$ we define the space $G^s_{\tau, r}(\mathbb{T}^n)$ of all the (n-tuples of) functions satisfying

$$\sum_{\alpha \in \mathbb{Z}^n} |\alpha|^{2r} e^{2\tau|\alpha|^{1/s}} |u_\alpha|^2 < \infty. \tag{2.5}$$

Note that $G^s_{\tau, r}(\mathbb{T}^n) \subset (G^s(\mathbb{T}^n))^n$ and for every fixed $r \geq 0$

$$\bigcup_{\tau > 0} G^s_{\tau, r}(\mathbb{T}^n) = (G^s(\mathbb{T}^n))^n.$$

The spaces $G^1_{\tau, r}(\mathbb{T}^n)$, $\tau > 0$, $r \geq 0$ are then subsets of the analytic class in $\mathbb{T}^n$. Finally, concerning B in (2.1), we have $B(u) = b(u, u)$ where $b(u, v)$ is defined as usual by

$$(b(u, v), w) = \sum_{j, k=1}^{n} \int_{\mathbb{T}^n} u^{(j)} \frac{\partial v^{(k)}}{\partial x_j} w^{(k)} dx,$$

which one can easily re-interpret in terms of the Fourier coefficients in (2.2).

Let us now state the result of Gevrey regularity for the solutions of (2.1). We shall prescribe initial data u_0 in $D(A^{1/2}) \subset H$, cf. (2.4). It is then well known that in dimension $n = 2$ the solution $u = u(t)$ exists and remains bounded in $D(A^{1/2})$ for all positive times, whereas for $n = 3$ this is granted only in a finite interval $[0, T]$. For all t in the domain of $D(A^{1/2})$ existence, we have:

Theorem 2.1. *Let the right-hand side f in* (2.1) *be given in* $(G^s(\mathbb{T}^n))^n \cap H$, $s \geq 1$. *Then the above solution* $u(t)$ *is an analytic function of* t, *with values in* $(G^s(\mathbb{T}^n))^n$.

More precisely, assume $f \in G^s_{\tau, 0}(\mathbb{T}^n) \cap H$, $s \geq 1$, $\tau > 0$. For small values of $t > 0$ we have $u(t) \in G^s_{t, 1}(\mathbb{T}^n) \cap H$, whereas for large values of t we get $u(t) \in G^s_{\sigma, 1}(\mathbb{T}^n) \cap H$, for a suitable constant σ, $\sigma \leq \tau$.

Let us observe that for $f \equiv 0$, or $f \in (G^1(\mathbb{T}^n))^n \cap H$, from Theorem 2.1 we deduce analyticity of the solution with respect to space and time variables, for $t > 0$.

3. Gevrey Hypoellipticity of Perturbations of Powers of the Schrödinger Operator

This Section deals with the properties of hypoellipticity and Gevrey hypoellipticity of perturbations of powers of the Schrödinger operator. The main properties of the operators depend heavily on the lower order terms of their symbol. In particular we consider the following class of linear differential operators:

$$P(x, y, D_x, D_y) = (D_y^2 - D_x)^p + \sum_{(l, j) \in I} b_{lj}(x, y) D_x^l D_y^j, \tag{3.1}$$

where $D_x = -i\dfrac{\partial}{\partial x}$ and $D_y = -i\dfrac{\partial}{\partial y}$; with $p, l, j \in \mathbb{Z}_+$, $p \geq 2$, and $b_{lj} : \Omega \to \mathbb{C}$ are smooth functions. The subset of the indices I corresponding to lower order terms will be described in the sequel. The respective class of symbols is the following:

$$p(x, y, \xi, \eta) = (\eta^2 - \xi)^p + \sum_{(l, j)\in I} b_{lj}(x, y)\xi^l \eta^j, \tag{3.2}$$

obtained from (3.1) by replacing the couple (ξ, η) to the couple of derivatives (D_x, D_y). We denote by $z = (x, y)$ the real variables in Ω, open subset in $\mathbb{R}^2$, neighborhood of a point z_0; $\zeta = (\xi, \eta)$ the dual variables of z.

To investigate (3.1), we follow closely the arguments in De Donno and Oliaro [4], where the class of operators

$$D_y^m - a(x, y)D_x^d + \sum_{(l, j)\in I} b_{lj}(x, y)D_x^l D_y^j, \ \ d < m$$

is studied in the anisotropic frame; also see De Donno and Rodino [5] about isotropic case and pseudo-differential coefficients. Hypoellipticity and solvability in [4], [5] depend only on lower order terms corresponding to a unique couple $(l^*, j^*) \in \mathbb{Z}_+^2$ with $k^* := \frac{m}{d} l^* + j^* > m - \frac{1}{2}$ $(k^* > m - 1,$ in the case of constant coefficients), such that $\operatorname{Im} b_{l^* j^*}(x, y) \neq 0$. We also obtained a sufficient condition for G^s hypoellipticity, $s \geq \frac{1}{k^* - (m-1)}$ in [4], $s \geq \frac{m/d}{k^* - (m-1)}$ in [5] (cf. the definition of Gevrey spaces in (1.2)). On the contrary, for P in (3.1), terms of much lower order may play a role, see the next Theorem 3.1. It is also natural to allow more couples (l_i^*, j_i^*), $i = 1, \dots, n$, to give the order $k^* = 2l_i^* + j_i^*$. The case of p-powers of generic operators $(D_y^m - a(x, y)D_x^d)$ shall be detailed in a future paper.

Let us observe that for operators of the form

$$(D_y^2 - D_x)^p + iD_x^{l^*} D_y^{j^*}, \tag{3.3}$$

we shall prove in this Section that $k^* := 2l^* + j^*$ varying in the interval

$(p, 2p)$ is allowed if $p \geq 2$; while in the case $p = 1$ we obtained in [4], [5] and [15] that the lower order terms have no influence.

We define the following sets for $k \in \mathbb{Q}_+$, $0 < k < 2p$:

$$I_k = \{(l, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : 2l + j = k\}$$

and fix $k = k^*$ such that $p < k^* < 2p$. We use the notations k^- for all $k < k^*$, k^+ for all $k > k^*$. We define $I = I_- \cup I_{k^*} \cup I_+$, with $I_- = \cup I_{k^-}$ and $I_+ = \cup I_{k^+}$.

In this Section we prove $S^m_{\rho,\delta}$ estimates of (3.2) in order to obtain hypoellipticity and Gevrey hypoellipticity of the class of operators (3.1).

In the plane (ξ, η) we consider the conic neighborhood $\Lambda = \{|\eta| < C\xi\}$ of the semi-axis $\xi > 0$, for a suitable constant C; we observe that the intersection of the half-plane $\xi < 0$ with the curve $\eta^2 - \xi = 0$ is empty.

We denote with Γ the set $\Omega \times \Lambda$, and we state the following:

Theorem 3.1. *Assume that* I_{k^*} *consists of couples* $(l_i^*, j_i^*), i = 1, \ldots, n$, $k^* = 2l_i^* + j_i^*$ *with* $p + 1 \leq k^* < 2p$ *such that*:

(i) $\sum_{i=1}^n \operatorname{Im} b_{l_i^* j_i^*}(x, y)\xi^{l_i^*}\eta^{j_i^*} \neq 0$, *for all* $(z, \zeta) \in \Gamma$, $\eta \neq 0$;

(ii) $b_{lj}(x, y) \equiv 0$ *for all* $(l, j) \in I_{k^+}$, $(2l + j > k^*)$;

(iii) $\sum_{i=1}^n \operatorname{Re} b_{l_i^* j_i^*}(x, y)\xi^{l_i^*}\eta^{j_i^*} \geq 0$, *for even* p,

$$\sum_{i=1}^n \operatorname{Re} b_{l_i^* j_i^*}(x, y)\xi^{l_i^*}\eta^{j_i^*} \equiv 0, \text{ for odd } p.$$

Then

$$|p(x, y, \xi, \eta)| \geq b|\zeta|^{\frac{1}{2}k^*} \quad \text{in } \Omega \times \mathbb{R}^2,$$

for a suitable constant $b > 0$, *and for all* $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, $\beta = (\beta_1, \beta_2)$

$\in \mathbb{Z}_+^2$ *and for all* $K \subset\subset \Omega$ *we have with suitable constants* $L_{\alpha,\beta}$ *and* B *that*:

$$\frac{|D_x^{\alpha_1} D_y^{\alpha_2} D_\xi^{\beta_1} p(x, y, \xi, \eta)| |\zeta|^{\rho|\alpha|-\delta|\beta|}}{|p(x, y, \xi, \eta)|} \le L_{\alpha,\beta}, \quad |\xi| + |\eta| > B, \tag{3.4}$$

where

$$\rho = \frac{k^* - p}{2p}, \quad \delta \ge 0, \tag{3.5}$$

for any $\delta < \rho$. *Since we have assumed* $p + 1 \le k^* < 2p$, *we have* $0 < \frac{1}{2p} \le \rho < \frac{1}{2}$.

Remark 3.2. By formula (3.4) and by Mascarello and Rodino [14, Theorem 3.3.6], we have that an operator $P(z, D_z)$, associated to the symbol $p(z, \zeta)$ in (3.2), is C^∞-microlocally hypoelliptic in Γ; i.e., $\Gamma \cap WFPu = \Gamma \cap WFu$, for all $u \in \mathcal{D}'(\Omega)$, where WFu is the Hörmander wave front set. A microhypoelliptic operator is hypoelliptic too.

Remark 3.3. If the coefficients are analytic, formula (3.4) holds for $L_{\alpha\beta} = L^{|\alpha|+|\beta|+1}\alpha!\beta!$, so by Kajitani and Wakabayashi [11, Theorem 1.9], we have that an operator $P(z, D_z)$, associated to the symbol $p(z, \zeta)$ in (3.2) is G^s-microlocally hypoelliptic in Γ for $s \ge \max\left\{\frac{1}{\rho}, \frac{1}{1-\delta}\right\} = \frac{1}{\rho}$, that is, $s \ge \frac{2p}{k^* - p}$.

As examples of operators satisfying Theorem 3.1, beside (3.3), we can consider the following:

Example.

$$P(x, y, D_x, D_y) = (D_y^2 - D_x)^p + i(y^2 + 1)D_x^l D_y^{p+h-2l},$$

having $k^* = p + h, 1 \le h \le p - 1$. They are hypoelliptic and G^s hypoelliptic for $s \ge \frac{2p}{h}$ in a neighborhood of the origin.

Example.

$$P(x,\ y,\ D_x,\ D_y) = (D_y^2 - D_x)^4 + iy^2 D_x D_y^4 + (x^2 + i(y+1))D_x^2 D_y^2 + (x+1)D_x^3,$$

where $k^* = 6$, the imaginary part is given by $\xi\eta^2(y^2\eta^2 + (y+1)\xi) \neq 0$ in $\Omega \times \Lambda$, $\eta \neq 0$, with Ω neighborhood of the origin and $\xi > 0$; and we also have $\xi^2(x^2\eta^2 + (x+1)\xi) \geq 0$. We obtain hypoellipticity and G^s hypoellipticity for $s \geq 4$.

Remark 3.4. When $\rho < 1$, and $\delta > 0$, one can prove by means of interpolation theory as in Wakabayashi [22, Theorem 2.6] that (3.4) is valid for any $(\alpha, \beta) \in \mathbb{Z}_+^4$, if (3.4) holds for $|\alpha| + |\beta| = 1$. Hence it is sufficient to verify (3.4) for $|\alpha| + |\beta| = 1$ since we can consider $\delta > 0$ in Theorem 3.1.

Proof of Theorem 3.1. First we estimate the numerator of (3.4) and then we give some lemmas to estimate the denominator, see Lemma 3.5, 3.7 and 3.8. In view of hypothesis (ii) in Theorem 3.1 we have $I_+ \equiv 0$, so for $\alpha_1 = 1$, we get

$$|D_x p(x, y, \zeta)||\zeta|^{-\delta} = \left|\sum_{(l,j)\in I_{k^*}\cup I_-} D_x b_{lj}(x, y)\xi^l\eta^j\right||\zeta|^{-\delta} \leq L_1\left(\sum_{(l,j)\in I_{k^*}\cup I_-}|\xi|^l|\eta|^j\right)|\zeta|^{-\delta};$$

and similarly for $\alpha_2 = 1$

$$|D_y p(x,\ y,\ \zeta)||\ \zeta\ |^{-\delta} \leq L_2\left(\sum_{(l,\ j)\in I_{k^*}\cup I_-}|\ \xi\ |^l|\ \eta\ |^j\right)|\ \zeta\ |^{-\delta},$$

for suitable constants L_1, L_2. If $\beta_1 = 1$,

$$|D_\xi p(z,\ \xi,\ \eta)||\ \zeta\ |^\rho \leq L_3\left(|\eta^2 - \xi|^{p-1} + \sum_{(l,\ j)\in I_{k^*}\cup I_-}|\ \xi\ |^{l-1}|\ \eta\ |^j\right)|\ \zeta\ |^\rho;$$

and for $\beta_2 = 1$

$$|D_\eta p(z, \xi, \eta)||\zeta|^\rho \le L_4\left(|\eta^2 - \xi|^{p-1}|\eta| + \sum_{(l, j)\in I_{k^*}\cup I_-}|\xi|^l|\eta|^{j-1}\right)|\zeta|^\rho,$$

with suitable constants L_3, L_4.

To prove (3.4), it will be then sufficient to show the boundedness, for $|\zeta| > B$, of the functions

$$Q_1(z, \zeta) = \frac{\left(\sum_{(l, j)\in I_{k^*}\cup I_-}|\xi|^l|\eta|^j\right)}{|p(z, \zeta)|},$$

$$Q_2(z, \zeta) = \frac{\left(|\eta^2 - \xi|^{p-1}|\eta| + \sum_{(l, j)\in I_{k^*}\cup I_-}|\xi|^l|\eta|^j\right)|\zeta|^\rho}{|p(z, \zeta)|},$$

$$Q_3(z, \zeta) = \frac{\left(|\eta^2 - \xi|^{p-1} + \sum_{(l, j)\in I_{k^*}\cup I_-}|\xi|^{l-1}|\eta|^j\right)|\zeta|^\rho}{|p(z, \zeta)|}.$$

First we introduce three regions in the plane (ξ, η):

$$R_1 : c\xi \le \eta^2 \le C\xi$$

$$R_2 : \eta^2 \ge C\xi$$

$$R_3 : \eta^2 \le c\xi, \tag{3.6}$$

for suitable constants c, C satisfying the inequalities $c << \frac{1}{2}$, and $C >> 2$, cf. [4, 5].

The following estimates then hold:

$$|\zeta|^{-\delta} \le \begin{cases}\text{const.}|\eta|^{-2\delta}, & \zeta \in R_1\\ \text{const.}|\eta|^{-\delta}, & \zeta \in R_2\\ \text{const.}\xi^{-\delta}, & \zeta \in R_3;\end{cases} \tag{3.7}$$

and

$$|\zeta|^{\rho} \leq \begin{cases} \text{const.}|\eta|^{2\rho}, & \zeta \in R_1 \\ \text{const.}|\eta|^{2\rho}, & \zeta \in R_2 \\ \text{const.}\xi^{\rho}, & \zeta \in R_3. \end{cases} \tag{3.8}$$

By abuse of notation, in the following we shall also denote by R_1, R_2, R_3 the sets $\Omega \times R_1$, $\Omega \times R_2$, $\Omega \times R_3$; recall that $\Gamma = \Omega \times \Lambda$.

The following three lemmas give us some relevant estimates from the below of $|p(z, \zeta)|$ in (3.2).

Lemma 3.5. *Let* $p(z, \zeta)$ *be the function* (3.2), *such that* (i), (ii) *and* (iii) *in Theorem* 3.1 *hold. Then there are positive constants* $K_1 < 1$, B, *such that, for* $(z, \zeta) \in R_1$, $|\zeta| > B$.

$$|p(z, \zeta)| \geq K_1 \left((\eta^2 - \xi)^{2p} + \left(\sum_{i=1}^{n} \operatorname{Im} b_{l_i^* j_i^*}(x, y) \xi^{l_i^*} \eta^{j_i^*} \right)^2 \right)^{\frac{1}{2}}. \tag{3.9}$$

Proof. We have that

$$|p(z, \zeta)|^2 = \left((\eta^2 - \xi)^p + \sum\nolimits_{(l, j) \in I_{k^*} \cup I_-} \operatorname{Re} b_{lj}(x, y) \xi^l \eta^j \right)^2$$

$$+ \left(\sum\nolimits_{i=1}^{n} \operatorname{Im} b_{l_i^* j_i^*}(x, y) \xi^{l_i^*} \eta^{j_i^*} + \sum\nolimits_{(l, j) \in I_-} \operatorname{Im} b_{lj}(x, y) \xi^l \eta^j \right)^2. \tag{3.10}$$

By developing (3.10) and removing the terms

$$\left(\sum_{(l, j) \in I_{k^*} \cup I_-} \operatorname{Re} b_{lj}(x, y) \xi^l \eta^j \right)^2 \quad \text{and} \quad \left(\sum_{(l, j) \in I_-} \operatorname{Im} b_{lj}(x, y) \xi^l \eta^j \right)^2$$

respectively from the real and imaginary part of $p(z, \zeta)$, we can write

$$|p(z, \zeta)|^2 \geq (\eta^2 - \xi)^{2p} + \left(\sum_{i=1}^{n} \operatorname{Im} b_{l_i^* j_i^*}(x, y) \xi^{l_i^*} \eta^{j_i^*} \right)^2 + \sum_{i=1}^{3} J_i(z, \zeta),$$

where

$$J_1(z,\ \zeta) = 2(\eta^2 - \xi)^p \sum_{(l_i^*,\, j_i^*)\in I_{k^*}} \operatorname{Re} b_{l_i^*,\, j_i^*}(x,\ y)\xi^{l_i^*}\eta^{j_i^*} \tag{3.11}$$

$$J_2(z,\ \zeta) = \sum_{i=1}^{n} \operatorname{Im} b_{l_i^* j_i^*}(x,\ y)\xi^{l_i^*}\eta^{j_i^*} \sum_{(l,\, j)\in I_-} \operatorname{Im} b_{lj}(x,\ y)\xi^l\eta^j, \tag{3.12}$$

$$J_3(z,\ \zeta) = 2(\eta^2 - \xi)^p \sum_{(l,\, j)\in I_-} \operatorname{Re} b_{lj}(x,\ y)\xi^l\eta^j. \tag{3.13}$$

The function (3.11) is non-negative by hypothesis (iii). Let us fix attention on $J_2(z,\ \zeta)$ and $J_3(z,\ \zeta)$ defined respectively by (3.12) and (3.13). We have for all $\varepsilon > 0$

$$\frac{1}{2}\left(\sum_{i=1}^{n} \operatorname{Im} b_{l_i^* j_i^*}(x,\ y)\xi^{l_i^*}\eta^{j_i^*}\right)^2 + J_2(z,\ \zeta) \geq \left(\frac{1}{2} - \varepsilon\right)\left(\sum_{i=1}^{n} \operatorname{Im} b_{l_i^* j_i^*}(x,\ y)\xi^{l_i^*}\eta^{j_i^*}\right)^2,$$

in $R_1, |\zeta| > B$. In fact by (3.6) in R_1 and hypothesis (i) in Theorem 3.1, for all $\varepsilon > 0$ we get for B sufficiently large

$$\frac{|J_2(z,\ \zeta)|}{\left(\sum_{i=1}^{n} \operatorname{Im} b_{l_i^* j_i^*}(x,\ y)\xi^{l_i^*}\eta^{j_i^*}\right)^2} \leq \text{const.} \sum_{(l,\, j)\in I_-} \frac{|\eta|^{k^*+2l+j}}{\eta^{2k^*}} < \varepsilon,\ \ |\zeta| > B.$$

We remark that $k^* = 2l^* + j^* > k^- = 2l + j$. Concerning (3.13) we also have that:

$$\frac{1}{2}(\eta^2 - \xi)^{2p} + \frac{1}{2}\left(\sum_{i=1}^{n} \operatorname{Im} b_{l_i^* j_i^*}(x,\ y)\xi^{l_i^*}\eta^{j_i^*}\right)^2 + J_3(z,\ \zeta)$$

$$\geq \left(\frac{1}{2} - \varepsilon\right)(\eta^2 - \xi)^{2p} + \left(\sum_{i=1}^{n} \operatorname{Im} b_{l_i^* j_i^*}(x,\ y)\xi^{l_i^*}\eta^{j_i^*}\right)^2\Bigg]$$

in $R_1, |\zeta| > B$, since

$$\frac{|J_3(z,\ \zeta)|}{(\eta^2-\xi)^{2p}+\left(\sum_{i=1}^{n}\operatorname{Im} b_{l_i^* j_i^*}(x,\ y)\xi^{l_i^*}\eta^{j_i^*}\right)^2}<\varepsilon,\ |\zeta|>B. \tag{3.14}$$

In fact, considering the curves $\xi=(1-r)\eta^2$ in the plane $(\xi,\ \eta)$, $|r|<1$, we obtain by (3.6) in R_1 that the left part of (3.14) is estimated by

$$\sum_{(l,j)\in I_-}\frac{r^p|\eta|^{2p+2l+j}}{r^{2p}|\eta|^{4p}+\eta^{2k^*}}\leq\sum_{(l,j)\in I_-}\frac{t^p|\eta|^{2p+2l+j-2k^*}}{|\eta|^{4p-2k^*}+t^{2p}}<\varepsilon,\ |t|+|\eta|>B, \tag{3.15}$$

where $t=\frac{1}{r}$, by dividing for η^{k^*}; since $k^-<k^*$. Recall that:

For all $\varepsilon>0$ there exists $B_\varepsilon>0$ such that for $|x|+|y|>B_\varepsilon$,

$$\frac{x^\alpha y^\beta}{x^{2\gamma}+y^{2\nu}}<\varepsilon \text{ if and only if } (2\gamma-\alpha)(2\nu-\beta)>\alpha\beta.$$

For $|r|\geq 1$, in R_1, we have

$$\sum_{(l,\ j)\in I_-}\frac{t^p|\eta|^{2p+2l+j-2k^*}}{|\eta|^{4p-2k^*}+t^{2p}}\leq|\eta|^{-2p+2l+j}<\varepsilon,\ \text{since } k^-<2p. \tag{3.16}$$

Then

$$|p(z,\ \zeta)|\geq K_1\left((\eta^2-\xi)^{2p}+\left(\sum_{i=1}^{n}\operatorname{Im} b_{l_i^* j_i^*}(x,\ y)\xi^{l_i^*}\eta^{j_i^*}\right)^2\right)^{\frac{1}{2}},|\zeta|>B. \tag{3.17}$$

for a suitable positive constant K_1.

Remark 3.6. From the previous estimate (3.17) follows easily that $|p(z,\ \zeta)|\geq|\zeta|^{\frac{1}{2}k^*}$.

Lemma 3.7. *Let* $p(z,\ \zeta)$ *be the function* (3.2). *Then there are positive constants* $K_2<1$, B, *such that*:

$$|p(z,\ \zeta)|\geq K_2|\eta|^{2p},\ (z,\ \zeta)\in R_2,|\zeta|>B. \tag{3.18}$$

Lemma 3.8. *Let* $p(z, \zeta)$ *be the function* (3.2). *Then there are positive constants* $K_3 < 1$, B, *such that*:

$$| p(z, \zeta) | \geq K_3 | \xi |^p, \ (z, \zeta) \in R_3, | \zeta | > B. \tag{3.19}$$

For the proofs of Lemma 3.7 and Lemma 3.8 see De Donno and Rodino [5] concerning similar operators, the present case actually does not involve more complications.

By the relations (3.6), (3.7) and (3.8), and the inequality $|\eta^2 - \xi| \leq | \eta |^2 + | \xi |$, we easily estimate the numerators N_i, $i = 1, 2, 3$, of Q_i, in the regions R_2 and R_3 in the following way:

$$N_1(\zeta) \leq \begin{cases} \text{const.} | \eta |^{2l+j-\delta}, & \zeta \in R_2 \\ \text{const.} | \xi |^{l+\frac{1}{2}j-\delta}, & \zeta \in R_3; \end{cases} \tag{3.20}$$

$$N_2(\zeta) \leq \begin{cases} \text{const.} | \eta |^{2p-1+2\rho}, & \zeta \in R_2 \\ \text{const.} | \xi |^{p-\frac{1}{2}+\rho}, & \zeta \in R_3; \end{cases} \tag{3.21}$$

$$N_3(\zeta) \leq \begin{cases} \text{const.} | \eta |^{2p-2(1-\rho)}, & \zeta \in R_2 \\ \text{const.} | \xi |^{p-1+\rho}, & \zeta \in R_3. \end{cases} \tag{3.22}$$

We have $N_3(z, \zeta) \leq N_2(z, \zeta)$ in R_2 and R_3, so we can just consider the functions $N_1(z, \zeta)$ and $N_2(z, \zeta)$ in those regions. Now Lemma 3.7, Lemma 3.8 and the estimates (3.20) and (3.21) show the boundedness of $Q_1(z, \zeta)$ and $Q_2(z, \zeta)$ in the regions R_2 and R_3 since in (3.5) always it is $\rho < \frac{1}{2}$; so for $Q_3(z, \zeta)$, too. For $Q_1(z, \zeta)$ in R_1 we have immediately boundedness since we apply Remark 3.6, formula (3.8) in R_1, recalling that in Q_1, $(l, j) \in I_{k^*} \cup I_-$ with $\delta \geq 0$.

Regarding Q_2 and Q_3 in the region R_1, we observe that $|\eta^2 - \xi|$ vanishes in it, so estimates of the previous type are not optimal.

For Q_2, by (3.6), (3.8) and (3.9) we get easily:

$$Q_2(z, \zeta) \le \text{const.} \frac{|\eta^2 - \xi|^{p-1}|\eta|^{1+2\rho}}{\left((\eta^2 - \xi)^{2p} + \left(\sum_{i=1}^{n} \operatorname{Im} b_{l_i^* j_i^*}(x, y)\xi^{l_i^*}\eta^{j_i^*}\right)^2\right)^{\frac{1}{2}}}$$

$$+ \sum_{(l, j)\in I_{k^*}\cup I_-} \frac{|\eta|^{2l+j-1+2\rho}}{|\eta|^{k^*}} \le L, \ (x, y) \in \Omega, \ |\zeta| > B.$$

The second term in the right-hand side is bounded since $\rho < \frac{1}{2}$ by (3.5). About the first term we can argue in the same way we have done in Lemma 3.5, see formulas (3.14), (3.15) and (3.16), obtaining $\rho \le \frac{k^* - p}{2p}$.

The study of the boundedness of the functions $Q_3(z, \zeta)$ in the region R_1 actually does not involve further complicated statements, so arguing like the previous step and using the estimates (3.5) on ρ, we have proved that $Q_i(z, \zeta)$, $i = 1, 2, 3$ is also bounded in R_1.

The following Remark ends the proof:

Remark 3.9. By formulas (3.9), (3.18) and (3.19), we obtain that $|p(z, \zeta)| \ge b|\zeta|^{\frac{1}{2}k^*}$, $b > 0$, $|\zeta| > B$. In fact we obtain that $|p(z, \zeta)| \ge$ const.$|\eta|^{k^*}$ in R_1, then $|\eta|^{k^*} = \frac{1}{2}|\eta|^{k^*} + \frac{1}{2}|\eta|^{k^*} \ge \text{const.}(|\eta|^{k^*} + |\xi|^{\frac{1}{2}k^*})$ $\ge \text{const.}(|\eta|^{\frac{1}{2}k^*} + |\xi|^{\frac{1}{2}k^*}) \sim \text{const.}|\zeta|^{\frac{1}{2}k^*}$, so $|p(z, \zeta)| \ge b|\zeta|^{\frac{1}{2}k^*}$. In the same way we get $|p(z, \zeta)| \ge b|\zeta|^p$ in R_2 and R_3, the result follows since we have $k^* < 2p$.

4. Gevrey Local Solvability for the Square of the Schrödinger Operator with a Nonlinear Perturbation

In this Section we consider the following equation:

$$(D_y^2 - D_x)^2 u + F(x, y; \partial_x^l \partial_y^j u)|_{2l+j<4} = \mu f(x, y), \tag{4.1}$$

where $F(x, y; z), \in \mathbb{C}^6$, is a nonlinear function. Before giving the main result let us recall that, given an open set $\Omega \subset \mathbb{R}^2$, the Gevrey anisotropic space $G^{(\lambda_1, \lambda_2)}(\Omega), \lambda_1 \geq 1, \lambda_2 \geq 1$, is the set of all C^∞ functions such that for every compact set $K \subset \Omega$ there exists a positive constant C_K satisfying $\sup_K |\partial_x^l \partial_y^j f(x, y)| \leq C_K^{l+j+1} l!^{\lambda_1} j!^{\lambda_2}$ for every $l, j \in \mathbb{Z}_+$. As standard $G_0^{(\lambda_1, \lambda_2)}(\Omega), \lambda_1, \lambda_2 > 1$, is the set of all the functions in $G^{(\lambda_1, \lambda_2)}(\Omega)$ with compact support in Ω.

We shall prove the following result.

Theorem 4.1. *Let us fix* $\lambda = (\lambda_1, \lambda_2), 1 < \lambda_1 < 4, \lambda_2 > 1$. *We suppose that the datum* f *in* (4.1) *is in* $G_0^\lambda(\Omega_\delta), \Omega_\delta := \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < \delta\}$. *Moreover, rewriting* $F(x, y; z)$ *as* $G(x, y; \Re z; \Im z)$ *and setting* $w = (\Re z, \Im z)$ *we suppose that:*

$G(x, y; 0) = 0;$

$G(x, y; w_0) \in G^\lambda(\Omega_\delta)$ *for every* $w_0 \in \mathbb{R}^{12}$;

$G(x_0, y_0; w) \in G^\sigma(\mathbb{R}^{12})$, *for every* $(x_0, y_0) \in \Omega_\delta, \sigma < 4$.

Then the semilinear equation (4.1) *admits a classical solution in* Ω_δ, *for* δ *and* μ *sufficiently small.*

Remark 4.2. Arguing in the isotropic case we have from the previous theorem the $G^{4-\varepsilon}$ local solvability of (4.1) for every $\varepsilon > 0$.

We want now to recall the definition of a class of Gevrey-Sobolev spaces on the strip $\mathbb{R} \times (-\delta, \delta), \delta > 0$. These spaces have been studied in the isotropic form by Gramchev and Rodino [9] and then generalized to the anisotropic case in Marcolongo and Oliaro [13], see also De Donno and Oliaro [4]. To start with, let us recall that the space $H_p^s(\mathbb{R} \times (-\delta, \delta))$, $p < 1$, s integer, is the set of the L^2 functions satisfying

$$\| f(x, y) \|_{H_p^s}^2 := \sum_{k=0}^{s} \int_{-\delta}^{\delta} \int_{-\infty}^{\infty} (1 + |\xi|^{2p})^{s-k} |D_y^k \widetilde{f}(y, \xi)|^2 d\xi dy < \infty, \quad (4.2)$$

where $\widetilde{f}(y, \xi) = \int e^{-ix\xi} f(x, y)dx$ is the Fourier transform of f with respect to x. This definition extends to every $s > 0$ by interpolation. We fix now $s > 0$, $p < 1$, $\tau > 0$ and $r = \frac{1}{2} + \varepsilon$, for a fixed (arbitrarily small) ε; we set $q = \frac{1}{p}$.

Definition 4.3. Let us suppose that $\psi(y, \xi)$ is a positive function belonging to the Hörmander class $S_{1,0}^{pr}((-\delta, \delta)\times\mathbb{R})$. We define the Gevrey-Sobolev space $\mathbb{H}_{\tau, q, r}^{s, \psi}(\mathbb{R} \times (-\delta, \delta))$ as the set of all functions $f \in L^2(\mathbb{R} \times (-\delta, \delta))$ such that $\| f \|_{s, q}$ is finite, where

$$\| f \|_{s, q} := \|e^{\tau\psi(y, D_x)} f(x, y)\|_{H_p^s(\mathbb{R}\times(-\delta, \delta))}; \tag{4.3}$$

the operator $e^{\tau\psi(y, D_x)}$ acts on the function f in the following way:

$$e^{\tau\psi(y, D_x)} f(x, y) = \frac{1}{2\pi}\int e^{ix\xi} e^{\tau\psi(y, \xi)} \widetilde{f}(y, \xi)d\xi.$$

Observe that the operator $e^{\tau\psi(y, D_x)}$ and its inverse $e^{-\tau\psi(y, D_x)}$ establish an isometry between Hilbert spaces:

$$e^{\tau\psi(y, D_x)} : \mathbb{H}_{\tau, q, r}^{s, \psi}(\mathbb{R} \times (-\delta, \delta)) \to H_p^s(\mathbb{R} \times (-\delta, \delta)) \tag{4.4}$$

$$e^{-\tau\psi(y, D_x)} : H_p^s(\mathbb{R} \times (-\delta, \delta)) \to \mathbb{H}_{\tau, q, r}^{s, \psi}(\mathbb{R} \times (-\delta, \delta)). \tag{4.5}$$

In the next theorem we summarize the most important properties of $\mathbb{H}_{\tau, q, r}^{s, \psi}(\mathbb{R} \times (-\delta, \delta))$.

Theorem 4.4. *Let us suppose that the function* $\psi(y, \xi)$ *satisfies the condition*

$$\psi(y, \xi_1) - \psi(y, \xi_1 - \xi_2) - \psi(y, \xi_2) \leq -b \min\{1 + |\xi_1 - \xi_2|^{1/q}, 1 + |\xi_2|^{1/q}\}^r \tag{4.6}$$

for a constant $b > 0$. *Then there exists* $s_{\text{alg}} > 0$ *such that, for every* $s > s_{\text{alg}}$, $\mathbb{H}_{\tau, q, r}^{s, \psi}$ *is an algebra.*

Let us fix $1 < \lambda_1 < \frac{q}{r}, \lambda_2 > 1$. Let $f \in G_0^{(\lambda_1, \lambda_2)}(\mathbb{R} \times (-\delta, \delta))$. Then, for every function ψ satisfying the assumptions of Definition 4.3 for every $\tau > 0$ and $s \geq 0$ we have that $f \in \mathbb{H}^{s, \psi}_{\tau, q, r}(\mathbb{R} \times (-\delta, \delta))$.

Let us consider now the linear part of the equation (4.1) $P = (D_y^2 + D_x)^2$: we want to analyze the following operator:

$$\widetilde{P} = e^{\tau\psi(y, D_x)} P e^{-\tau\psi(y, D_x)}, \tag{4.7}$$

where we choose from now on $\psi(y, \xi) = \left(1 + \frac{y}{2\delta}\right)\varphi(\xi)|\xi|^{r/2}$, φ being a C^∞ function satisfying $0 \leq \varphi(\xi) \leq 1$ for every ξ, $\varphi(\xi) \equiv 0$ for $|\xi| \leq \frac{1}{2}$, $\varphi(\xi) \equiv 1$ for $|\xi| \geq 1$ (observe that $\psi(y, \xi)$ satisfies (4.6)). The next proposition then follows from the results of De Donno and Oliaro [4].

Proposition 4.5. *The symbol of the operator* $e^{\tau\psi(y, D_x)}(D_y^2 - D_x)e^{-\tau\psi(y, D_x)}$ *is given by*:

$$(\eta^2 - \xi) + i\frac{\tau}{\delta}\varphi(\xi)|\xi|^{r/2}\eta + p_{1+r-\nu}(x, y, \xi, \eta), \tag{4.8}$$

where $p_{1+r-\nu}(x, y, \xi, \eta)$ *satisfies the following estimates in a neighborhood* Γ *of the set* $\overline{\Sigma} = \{(\xi, \eta) \in \mathbb{R}^2 \setminus (0, 0) : \eta^2 - \xi = 0\}$, Γ *being of the form* $\Gamma = \{(\xi, \eta) \in \mathbb{R}^2 \setminus (0, 0) : c_0^{-1}\eta^2 \leq |\xi| \leq c_0\eta^2\}$, $c_0 > 1$:

$$|D_x^l D_y^j D_\xi^k D_\eta^h p_{1+r-\nu}(x, y, \xi, \eta)| \leq C_{ljhk}(1 + |\xi|^{1/2} + |\eta|)^{(1+r-\nu)-2k-h},$$

for every $l, j, k, h \in \mathbb{Z}_+$.

The results in De Donno and Oliaro [4] assure us that the operator $e^{\tau\psi(y, D_x)}(D_y^2 - D_x)e^{-\tau\psi(y, D_x)}$ is C^∞ microlocally hypoelliptic in Γ, in particular, denoting its symbol by $q(x, y, \xi, \eta)$, we have:

$$|q(x, y, \xi, \eta)| \geq c(|\xi|^{1/2} + |\eta|)^{1+r},$$

$$\frac{|D_x^l D_y^j D_\xi^k D_\eta^h q(x,\ y,\ \xi,\ \eta)|\,(|\,\xi\,|^{1/2} + |\,\eta\,|)^{r(2k+h)-(1-r)(2l+j)}}{|\,q(x,\ y,\ \xi,\ \eta)\,|} \le L_{ljkh},$$

for every $l,\ j,\ k,\ h \in \mathbb{Z}_+$ and $|\,\xi\,| + |\,\eta\,| >> 0$.

Now we observe that, since $e^{\tau\psi(y, D_x)} e^{-\tau\psi(y, D_x)} = e^{-\tau\psi(y, D_x)} e^{\tau\psi(y, D_x)} = Id$ we can write $\widetilde{P}$, cf. (4.7), in the following way:

$$\widetilde{P} = (e^{\tau\psi(y, D_x)}(D_y^2 - D_x)e^{-\tau\psi(y, D_x)})(e^{\tau\psi(y, D_x)}(D_y^2 - D_x)e^{-\tau\psi(y, D_x)}). \quad (4.9)$$

So we can apply Proposition 4.5 to the two factors in the right hand side of (4.9), obtaining that the symbol of the operator $\widetilde{P}$ equals, modulo lower order terms, the square of the (microlocally hypoelliptic in Γ) symbol (4.8). Then we can construct a microlocal parametrix of $\widetilde{P}$ in Γ. Since outside Γ the operator $\widetilde{P}$ is microlocally quasi-elliptic, a standard technique of patching together the microlocal parametrices in Γ and outside Γ (cf. Gramchev and Rodino [9], Marcolongo and Oliaro [13]), gives us a parametrix $\widetilde{E}$ of $\widetilde{P}$ in the space $H^s_{1/2}(\mathbb{R} \times (-\,\delta,\ \delta))$:

$$\widetilde{E} : H^s_{1/2}(\mathbb{R} \times (-\,\delta,\ \delta)) \to H^{s+2(1+r)}_{1/2}(\mathbb{R} \times (-\,\delta,\ \delta)),$$

with

$$\widetilde{P}\widetilde{E}u = \widetilde{\chi}(x,\ y)u + \widetilde{R}u,$$

where $\widetilde{\chi}(x,\ y) \equiv 1$ in a neighborhood of $(0,\ 0)$ and $\widetilde{R}$ is a regularizing operator in $H^s_{1/2}(\mathbb{R} \times (-\,\delta,\ \delta))$. Then by (4.4), (4.5) and (4.7) we have the following result.

Proposition 4.6. *There exists a linear map*

$$E : \mathbb{H}^{s,\,\psi}_{\tau,\,2,\,r}(\mathbb{R} \times (-\,\delta,\ \delta)) \to \mathbb{H}^{s+2(1+r),\,\psi}_{\tau,\,2,\,r}(\mathbb{R} \times (-\,\delta,\ \delta))$$

such that

$$PEu = \chi(x,\ y)u + Ru,$$

where $\chi(x, y) \in G_0^{(\lambda_1, \lambda_2)}(\Omega_\delta), 1 < \lambda_1 < \frac{2}{r}, \lambda_2 > 1, \chi(x, y) \equiv 1$ *in a neighborhood of the origin, and R is a regularizing operator in the space* $\mathbb{H}^{s,\psi}_{\tau, 2, r}$.

The nonlinearity can be treated with the technique developed by Bourdaud et al. [1]. The following proposition holds, cf. De Donno and Oliaro [4], Oliaro and Rodino [16].

Proposition 4.7. *Let us consider the nonlinearity in* (4.1); *we write for shortness* $J(u) = G(x, y, \Re(\partial_x^l \partial_y^j u), \Im(\partial_x^l \partial_y^j u))|_{2l+j<4}$. *We take* $u(x) \in \mathbb{B} \subset \mathbb{H}^{s,\psi}_{\tau,2,r}(\mathbb{R} \times (-\delta, \delta))$, *where* $\mathbb{B}$ *is bounded in* $\mathbb{H}^{s,\psi}_{\tau,2,r}$. *Then we can find a continuous non-decreasing function* $\Phi : (0, +\infty) \to (0, +\infty), \Phi(0) = 0$, *such that*

$$\| J(u) \|_{s, 2} \leq \Phi(\| u \|_{s+3, 2}); \tag{4.10}$$

moreover, if $u, v \in \mathbb{B}$ *we have*:

$$\| J(u) - J(v) \|_{s, 2} \leq C_{\mathbb{B}} \| u - v \|_{s+3, 2} \tag{4.11}$$

for every $s > s_{\text{alg}}$.

We can now prove the solvability of (4.1).

Proof of Theorem 4.1. Let us fix the datum $f \in \mathbb{H}^{s,\psi}_{\tau,2,r}$. Using Proposition 4.6 and arguments as in Gramchev and Popivanov [7], see also Gramchev and Rodino [9], Marcolongo and Oliaro [13], De Donno and Oliaro [4], we can find a positive, continuous, non-decreasing function $\mathcal{L} : [0, \delta_0] \to [0, +\infty)$, $\mathcal{L}(0) = 0$ such that, defining

$$\mathcal{A}_s(\delta) := \sup_{\substack{v \neq 0 \\ v \in \mathbb{H}^{s,\psi}_{\tau,2,r}(\Omega_\delta)}} \frac{\| Ev \|_{s+3, 2}}{\| v \|_{s, 2}}, \quad \mathcal{B}_s(\delta) := \sup_{\substack{v \neq 0 \\ v \in \mathbb{H}^{s,\psi}_{\tau,2,r}(\Omega_\delta)}} \frac{\| Rv \|_{s, 2}}{\| v \|_{s, 2}},$$

we have $\mathcal{A}_s(\delta) \leq \mathcal{L}(\delta)$, $\mathcal{B}_s(\delta) \leq \mathcal{L}(\delta)$. We are looking for a solution of the form $u = Ev$, so the equation (4.1) becomes $v(x) = \mathcal{Q}(v(x, y)) + \mu f(x, y)$, where $\mathcal{Q}(v(x, y)) := -Rv(x, y) - G(x, y; \Re(\partial_x^l \partial_y^j (Ev)(x, y)), \Im(\partial_x^l \partial_y^j (Ev)(x, y))|_{2l+j<4}$.

We then have to find a fixed point of the operator $\mathcal{Q}(\cdot) + f;$ we choose δ and μ such that the following conditions are satisfied:

$$\mathcal{B}_s(\delta)(1 + \mu\| f \|_{s,2}) + \Phi(\mathcal{A}_s(\delta)(1 + \mu\| f \|_{s,2})) \leq 1 \tag{4.12}$$

$$\mathcal{B}_s(\delta) + \mathcal{A}_s(\delta)C_{\mathbb{B}} < 1, \tag{4.13}$$

where $\Phi(\cdot)$ and $C_{\mathbb{B}}$ are the ones of Proposition 4.7, $\mathbb{B} := \{w \in \mathbb{H}^{s,\psi}_{\tau,2,r}(\Omega_\delta)$ $: \| w - \mu f \|_{s,2} \leq 1\}.$ Now, by (4.10) and (4.12) we have that $\mathcal{Q}(\cdot) + f : \mathbb{B} \to \mathbb{B};$ moreover (4.11) and (4.13) imply that $\mathcal{Q}(\cdot) + f$ is a contraction. We then obtain a solution as an application of the Fixed Point Theorem in the Banach space $\mathbb{B}.$ Taking s sufficiently large the solution is classical. By Theorem 4.4 and since $r = \frac{1}{2} + \varepsilon,$ with ε arbitrarily small, we obtain the solvability of (4.1) for $f \in G_0^\lambda(\Omega_\delta).$

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