

YAU'S THEOREM ON NILMANIFOLDS

Simon Salamon

A report on

[TW] V. Tosatti, B. Weinkove: The Calabi-Yau equation
on the Kodaira-Thurston manifolds, arXiv:0906.0634

TDGD, 17 July 2009

<http://calvino.polito.it/~salamon>

CONTENTS

- I Background
- II Special features
- III General theory
- IV Ricci tensors

PART I

BACKGROUND

Calabi conjecture (volume form)

I-1

Let (M^4, J, ω) be a compact Kähler surface, so that

- J is an integrable complex structure,
- ω is a symplectic form so $0 \neq [\omega] \in H^2(M, \mathbb{R})$.

Fix a smooth function F normalized so that

$$\int_M e^F \omega^2 = \int_M \omega^2 = 2 \operatorname{vol} M.$$

Theorem [Calabi, Aubin, Yau] There is a unique Kähler 2-form $\tilde{\omega}$ such that

$$\tilde{\omega}^2 = e^F \omega^2, \quad [\tilde{\omega}] = [\omega]$$

This problem still makes sense on an almost-Kähler 4-manifold when J is non-integrable.

The Kodaira-Thurston manifold

I-2

$M^4 = \mathbb{C}^2/\Gamma$ is a natural example arising from the classification of complex surfaces.

Here, $\mathbb{C}^2 = H \times \mathbb{R}$ is a nilpotent group, whereas

$$\Gamma = \Gamma' \times \mathbb{Z}$$

is a discrete subgroup acting holomorphically and preserving the symplectic form $dz_1 \wedge dz_2$.

$T^2 = Z_H \times S^1$ acts freely on M with quotient T^2 :

$$\begin{array}{ccc} M \cong \Gamma' \backslash H \times \mathbb{Z} \backslash \mathbb{R} \cong M' \times S^1 & & \\ & \pi \downarrow & \\ & T^2 & \end{array}$$

$b_1(M) = 3$, so M cannot admit a Kähler metric.

[K] K.Kodaira, Amer. J. Math. 86, 1964

linearly independent over \mathbf{R} . Hence the restriction to C of dw_2 is a non-vanishing holomorphic 1-form and its periods β_1, β_2 are linearly independent over \mathbf{R} . Consequently the representation $R: G \rightarrow R(G)$ is an isomorphism. $R(G)$ is a properly discontinuous group of affine transformations without fixed point of \mathbf{C}^2 and the quotient space $\mathbf{C}^2/R(G)$ is a (compact complex analytic) surface. $dw_1 \wedge dw_2$ is a holomorphic 2-form on S which does not vanish identically, while the canonical bundle of S is trivial. Hence $dw_1 \wedge dw_2$ does not vanish at each point of S . Therefore the pair of holomorphic functions w_1 and w_2 defines a locally biholomorphic map μ of W into \mathbf{C}^2 . The map μ induces a locally biholomorphic map of $S = W/G$ into $\mathbf{C}^2/R(G)$ which is surjective in view of the compactness of S . Hence μ is surjective and biholomorphic. Consequently S is complex analytically homeomorphic to $\mathbf{C}^2/R(G)$. Thus we obtain the following

THEOREM 19. *Let S be a surface. If the canonical bundle of S is trivial, then S is a K3 surface, a complex torus or an elliptic surface of the form \mathbf{C}^2/G , where \mathbf{C}^2 is the space of two complex variables (z_1, z_2) and G is a properly discontinuous group of affine transformations without fixed points of \mathbf{C}^2 which leave invariant the 2-form $dz_1 \wedge dz_2$. The first homology group of the elliptic surface \mathbf{C}^2/G is*

$$H_1(\mathbf{C}^2/G, \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_m,$$

where \mathbf{Z}_m denotes a cyclic group of a finite order m .

THEOREM 20. *Assume that the first Betti number b_1 of S is even and the geometric genus p_g of S is positive. If the first Chern class c_1 of S vanishes, then S is either a K3 surface or a complex torus.*

Proof. In case S has no meromorphic function except constants, S is, by Theorem 11, either a K3 surface or a complex torus, since the vanishing of c_1 implies that S contains no exceptional curve. We may assume therefore that S is either algebraic or elliptic. Since p_g is positive, S has an effective canonical divisor K . In view of the above Theorem 19, it suffices for our purpose to prove that K is zero. Since c_1 vanishes, K is homologous to zero. To prove that K is zero, it suffices therefore to show that every positive divisor on S is not homologous to zero. This is obvious if S is algebraic. Hence it suffices to consider the case in which S is a non-algebraic elliptic surface.

Let Δ denote the base curve of the elliptic surface S and let $\{E_j\}$ be a finite covering of Δ by small disks E_j . Since b_1 is even, there exists a

Left-invariant forms

I-3

The standard Riemannian metric g on M has a global basis of orthonormal 1-forms

$$e^1 = dx, \quad e^2 = dy, \quad e^3 = dz - xdy, \quad e^4 = dt.$$

where x, y are coordinates on $\pi(M) = T^2$.

As regards 2-forms, a basis

$$\omega^1 = e^{12} + e^{34}$$

$$\omega^2 = e^{13} + e^{42}$$

$$\omega^3 = e^{14} + e^{23},$$

of the subspace $\Lambda_+^2 T^*M$ gives rise to a triple

$$J_1$$

$$J_2$$

$$J_3,$$

of almost-complex structures with $J_1 = J_2 J_3$.

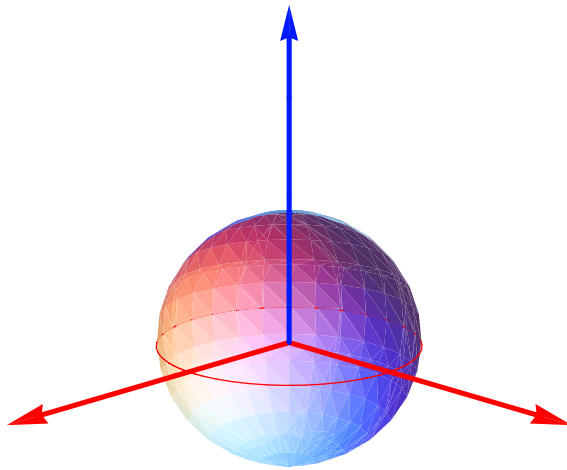
Spheres and moments

I-4

Lemma [Rend. Parma 17, 1991]

On any Riemannian 4-manifold,

- (a) J_2, J_3 integrable $\Rightarrow J_1$ integrable
- (b) J_2, J_3 symplectic $\Rightarrow J_1$ integrable
- (c) J_1 integrable, J_2 symplectic $\Rightarrow J_3$ symplectic



Here (b),(c) apply: J_1 is integrable and $\varpi = \omega_2 + i\omega_3$ is closed. Moreover, $\pi: M \rightarrow T^2$ is a 'moment map':

$$d(x+iy) = i \frac{\partial}{\partial t} \lrcorner \varpi.$$

PART II

SPECIAL FEATURES

Deforming the 2-form

II - 1

M continues to denote the Kodaira surface.

Fix attention on the almost-complex structure $J = J_3$. Any **T^2 -invariant 2-form**, cohomologous to $\omega = \omega_3$, compatible with J , equals

$$\tilde{\omega} = \omega + d\alpha, \quad \alpha = \sum_{i=1}^4 f_i(x, y) e^i.$$

Then

$$d\alpha = \sum_{i=1}^4 df_i \wedge e^i =$$

	+	-
	0	$a(e^{12} - e^{34})$
	0	$b(e^{13} - e^{42})$
	$f_{4,x}e^{14} + f_{4,y}e^{23}$	

with $a = f_3 - f_{2,x} + f_{1,y} = 0$, $b = f_{4,y} = f_{3,x}$, and

$$e^F = \tilde{\omega}^2 / \omega^2 = (1 + f_{4,x})(1 + f_{3,y}) - b^2$$

Negative chords

II - 2

Proposition [D] Any solution of

$$\tilde{\omega}^2 = e^F \omega^2, \quad [\tilde{\omega}] = [\omega]$$

compatible with J is unique.

Proof. Given two such solutions $\tilde{\omega}$ and $\tilde{\omega}' = \tilde{\omega} + d\beta$,

$$\begin{aligned} \tilde{\omega}^2 = (\tilde{\omega}')^2 &\Rightarrow 2\tilde{\omega} \wedge d\beta + d\beta^2 = 0 \\ &\Rightarrow \tau \wedge d\beta = 0, \end{aligned}$$

where $\tau = \tilde{\omega} + \tilde{\omega}'$ is also compatible with J . Relative to the associated metric,

$$d\beta \in \Lambda^2 T^*M$$

(so the 'chord' joining the two solutions is negative).
By Stokes'

$$0 = \int_M d\beta \wedge d\beta = - \int_M |d\beta|^2 \tau^2 \Rightarrow d\beta = 0.$$

[D] S.K. Donaldson, in *Inspired by S.S. Chern*, World Sci. 2006

The new metric

is defined by $\tilde{\omega}(\cdot, \cdot) = \tilde{g}(J\cdot, \cdot)$.

Write $\mathbb{R}^4 = V \oplus JV$ where $V = \langle e_1, e_2 \rangle = T(\text{base } T^2)$.
Relative to this splitting and basis,

$$\tilde{g} = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix},$$

where

$$X = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 + f_{4,x} & f_{4,y} \\ f_{3,x} & 1 + f_{3,y} \end{pmatrix}$$

resembles the first fundamental form of a surface.
Thus

$$u = \text{tr } X = \frac{1}{2} \text{tr } \tilde{g} = 2 + f_{4,x} + f_{3,y}$$
$$e^F = \det X = (1 + f_{4,x})(1 + f_{3,y}) - f_{3,x}^2.$$

A second 'fundamental matrix' is

$$\text{Hess } u = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}.$$

Lemma [TW]

$$\begin{aligned}\tilde{\Delta}u &= \sum_{i,j=1}^4 \tilde{g}^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} \\ &= \operatorname{tr} (X^{-1} \operatorname{Hess} u), \quad \text{by } T^2 \text{ invariance} \\ &= e^{-F} \left[(1 + f_{3,y}) u_{xx} - 2f_{3,x} u_{xy} + (1 + f_{4,x}) u_{yy} \right] \\ &= \Delta F + \sum_{k=2,4} \mathfrak{p}(X^{-1} \operatorname{Hess} f_k), \\ &\quad \text{where } \mathfrak{p}(Y) = (\operatorname{tr} Y)^2 - 2 \det Y \\ &\geq \Delta F.\end{aligned}$$

The crucial (and elegant) step is the penultimate one.

We next explain how this leads to the

Theorem [TW] Any solution of $\tilde{\omega}^2/\omega^2 = e^F$ satisfies

$$\sup_M \|u\| \lesssim \inf_M \Delta F.$$

PART III

GENERAL THEORY

Sobolev and parts

III – 1

$$\begin{aligned}
 \|u\|_{L^{2p}}^p &= \|u^{p/2}\|_{L^4}^2 \lesssim \|u^{p/2}\|_{W^{1,2}}^2 \\
 &= \int_M |d(u^{p/2})|^2 \omega^2 \\
 &= \int_M d(u^{p/2}) \wedge Jd(u^{p/2}) \wedge \omega \\
 &\sim \int_M u d(u^{p/2}) \wedge Jd(u^{p/2}) \wedge \tilde{\omega} \\
 &= -\frac{1}{4}p \int_M u^p d(Jdu) \wedge \tilde{\omega} \\
 &= -\frac{1}{8}p \int_M u^p (\tilde{\Delta}u) \tilde{\omega}^2 \\
 &\lesssim p \int_M u^p \omega^2 = p \|u\|_{L^p}^p
 \end{aligned}$$

A uniform bound

III - 2

Theorem [TW] Any solution of $\tilde{\omega}^2 = e^F \omega^2$ satisfies

$$\sup_M \|u\| \lesssim \inf_M \Delta F.$$

Proof. The previous inequality (to the $1/p$) is

$$\|u\|_{L^{2p}} \lesssim (1+p)^{1/p} \|u\|_{L^p}.$$

Iterating from $p = 1$,

$$\begin{aligned} \|u\|_{L^{2^k}} &\lesssim 2^{k/2^k} \dots 4^{1/4} 2^{1/2} \|u\|_{L^1} \\ &\rightarrow 4 \|u\|_{L^1}. \end{aligned}$$

Hence,

$$\begin{aligned} \|u\|_{C^0} &\lesssim \|u\|_{L^1} = \int_M u \omega^2 \\ &= \int_M 2\omega \wedge (\omega + d\alpha) \\ &= 4 \text{ vol } M. \end{aligned}$$

The continuity method

Fix $F \in C^\infty(T^2)$ with $\int \omega^2 = \int e^F \omega^2$. Set

$$c_t = \int \omega^2 / \int e^{tF} \omega^2,$$

and

$$\mathcal{I} = \left\{ t \in [0, 1] : \begin{array}{l} \tilde{\omega}_t^2 = c_t e^{tF} \omega^2 \\ [\tilde{\omega}_t] = [\omega] \end{array} \text{ is solvable in } [0, t] \right\}$$

Existing results in the literature will establish that \mathcal{I} is both open and closed. Therefore $\mathcal{I} = [0, 1]$.

We will have solved the original problem

$$\tilde{\omega}^2 = e^F \omega^2, \quad [\tilde{\omega}] = [\omega]$$

with a T^2 invariant $\tilde{\omega}$.

Closure

Theorem [W] \mathcal{T} is closed.

Idea of proof. Recall that

$$\|\tilde{\omega}_t\|_{C^0} \lesssim \|F\|_{C^2},$$

independently of t .

First one establishes a Hölder bound

$$\|\tilde{\omega}_t\|_{C^\alpha} \leq A,$$

for some $\alpha \in (0, 1)$. Bounds on the derivatives

$$\|\tilde{\omega}_t\|_{C^k} \leq A_k,$$

follow for each $k \geq 1$.

[W] B. Weinkove, JDG 76, 2007

To some extent this follows Yau's original proof of the Calabi conjecture.

[Y] S-T. Yau, CPAM, 1978

[J] D.D. Joyce, OUP books, 2000, 2007

Proposition [D] \mathcal{T} is open.

Idea of proof. Based on the elliptic complex

$$\Lambda^1 \xrightleftharpoons{d^*+d^+} \Lambda^0 \oplus \Lambda_+^2$$

modulo harmonic forms, and the implicit function theorem. Crucial to four dimensions.

Given a solution with $t = t_0$, we can first solve the volume equation with $[\tilde{\omega}_t] \in H^+$, where

$$H^2(M, \mathbb{R}) = H^+ \oplus H^- = \begin{array}{c|c} \begin{array}{c} + \\ e^{12} \end{array} & \begin{array}{c} - \\ e^{34} \end{array} \\ \hline [\omega_2] & [e^{13} - e^{42}] \\ \hline [\omega] = [\omega_3] & [e^{14} - e^{23}] \end{array}$$

In fact, $[\tilde{\omega}_t] = [\omega]$.

PART IV

RICCI TENSORS

Calabi conjecture (Ricci form)

IV – 1

Let (M, J, ω) be a compact Kähler surface. Its Ricci form

$$\rho = -i\partial\bar{\partial}\log(\det g_{\alpha\bar{\beta}})$$

represents minus the curvature of the *canonical line bundle* $\kappa = \Lambda^{2,0}M$, and $2\pi c_1(M) = [\rho]$. Moreover,

$$\begin{aligned}\tilde{\omega}^2/\omega^2 = e^F &\Rightarrow \log(\det \tilde{g}_{\alpha\bar{\beta}}) - \log(\det g_{\alpha\bar{\beta}}) = F \\ &\Rightarrow \tilde{\rho} - \rho = -i\partial\bar{\partial}F.\end{aligned}$$

Working backwards,

Theorem A 2-form $\tilde{\rho}$ satisfying $[\tilde{\rho}] = [\rho]$ is the Ricci form of a unique Kähler form $\tilde{\omega} = \omega + i\partial\bar{\partial}\phi$.

If $c_1(M) = 0$, we ‘obtain’ Ricci-flat metrics on M .

Assigning Ricci

IV – 2

Let M be the Kodaira surface with $J = J_3$. The Ricci and $*$ Ricci tensors of (M, g, J) are non-zero [A].

Consider instead the ‘Chern’ connection which has $\nabla\tilde{\omega} = 0 = \nabla J$, and torsion 2-forms in $\langle\omega_1, \omega_2\rangle$. Its Ricci form $\text{Ric}(\tilde{\omega}, J)$ is minus the curvature of $\kappa(J)$, now spanned by $\varpi = \omega_1 + i\omega_2$.

Example $\text{Ric}(\omega, J) = 0$, since $\nabla\varpi = -ie^1 \otimes \varpi$ for ω .

This time, starting from zero Ricci tensor, we obtain

Theorem [TW] Given $F \in C^\infty(T^2)$, there exists a T^2 invariant symplectic form $\tilde{\omega}$ compatible with J such that

$$\text{Ric}(\tilde{\omega}, J) = dJdF$$

[A] E. Abbena, BUMI 3, 1984