

# Il gruppo dei bi-simplettomorfismi degli spazi simmetrici Hermitiani \*

Antonio J. Di Scala †

*Alessandria, Maggio 2007*

---

\*In collaborazione con Andrea Loi e Guy Roos

†Politecnico di Torino

## The unit disc $\Delta \subset \mathbb{C}$ .

The unit disc  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$  has two well-known symplectic forms  $\omega_0$  and  $\omega_{hyp}$ :

$$\omega_0 = \frac{i}{2} dz \wedge d\bar{z},$$
$$\omega_{hyp} = \frac{\omega_0}{(1 - |z|^2)^2}$$

Let  $\mathcal{B}(\Delta)$  be the group of diffeomorphisms of  $\Delta$  such that:

$$f^*(\omega_0) = \omega_0,$$
$$f^*(\omega_{hyp}) = \omega_{hyp}.$$

The group  $\mathcal{B}(\Delta)$  is called *the group of bisymplectomorphisms of the disc  $\Delta$* .

Notice that the group  $U(1) \subset \mathcal{B}(\Delta)$ .

## The unit disc $\Delta \subset \mathbb{C}$ .

The following theorem characterizes the elements of  $\mathcal{B}(\Delta)$ .

**Theorem 0.1** *The elements  $f \in \mathcal{B}(\Delta)$  are the maps defined by*

$$f(z) = u(|z|^2) z \quad (z \in \Delta),$$

*where  $u$  is a smooth function  $u : [0, 1) \rightarrow S^1 \simeq U(1)$ .*

In other words, the restriction of a bisymplectomorphism  $f \in \mathcal{B}(\Delta)$  to a circle of radius  $r$  ( $0 < r < 1$ ) is the rotation  $u(r^2)$ .

Notice that if  $f \in \mathcal{B}(\Delta)$  then  $f(0) = 0$ .

# The unit disc $\Delta \in \mathbb{C}$ .

Sketch of the Proof of Theorem 0.1 :

- Since  $f$  preserves both symplectic forms then  $f$  preserves the quotient  $\frac{\omega_0}{\omega_{hyp}} = (1 - |z|^2)^2$ . Thus,

$$|f(z)| = |z|$$

for  $z \in \Delta$ .

- Conversely if  $|f(z)| = |z|$  for  $z \in \Delta$  and  $f$  preserves  $\omega_0$  then  $f \in \mathcal{B}(\Delta)$ .
- It is not difficult to show that the maps  $f(z) = u(|z|^2) z$  are bisymplectomorphisms.
- A simple computation shows that  $f(z) = v(|z|)z$  for  $z \in \Delta \setminus \{0\}$  and  $v : (0, 1) \rightarrow U(1)$  smooth.
- Whitney's Theorem can be used to show that  $v(|z|) = u(|z|^2)$  for a smooth  $u$ .  $\square$

# The bisymplectomorphism group of a bounded domain $\Omega \subset \mathbb{C}^n$ .

A bounded domain  $\Omega \subset \mathbb{C}^n$  has two well-known symplectic forms. Namely,

- The flat euclidean  $\omega_0$  :

$$\omega_0 = \frac{i}{2} dz \wedge d\bar{z},$$

- The Bergman form:

$$\omega_{\text{Berg}}(z) = \frac{i}{2} \partial \bar{\partial} \log K(z) ,$$

where  $K(z) := \sum_{j=1}^{\infty} \phi_j(z) \overline{\phi_j(z)}$  and  $\{\phi_j\}_{j=1}^{\infty}$  is an orthonormal basis in the space of holomorphic square integrable functions on  $\Omega$ .

Example: For the unit disc  $\Delta$  :  $K(z) = \frac{1}{\pi}(1 - |z|^2)^{-2}$ .

So,  $2\omega_{\text{hyp}} = \omega_{\text{Berg}}$ .

# The bisymplectomorphism group of a bounded domain $\Omega \subset \mathbb{C}^n$ .

**Definition 0.2** A bisymplectomorphism of  $\Omega$  is a diffeomorphism  $f : \Omega \rightarrow \Omega$  which satisfies

$$f^* \omega_0 = \omega_0, \tag{1}$$

$$f^* \omega_{\text{Berg}} = \omega_{\text{Berg}}, \tag{2}$$

that is, which preserves both symplectic forms.

Notice that  $f \in \mathcal{B}(\Omega)$  if and only if  $f$  preserves all linear combinations  $a \omega_0 + b \omega_{\text{Berg}}$ ,  $a, b \in \mathbb{R}$ . Thus,  $f$  also preserves the so called "hyperbolic metric"  $\omega_{\text{hyp}}$ .

The group  $\mathcal{B}(\Omega)$  is called *the group of bisymplectomorphisms of the bounded domain  $\Omega$* .

# The bisymplectomorphism group of a bounded domain $\Omega \subset \mathbb{C}^n$ .

Since both forms  $\omega_0$  and  $\omega_{\text{Berg}}$  are non degenerated there exists an operator  $B(z)$ , the Bergman operator, such that:

$$\omega_{\text{Berg}}(B(z)u, v) = \omega_0(u, v)$$

for  $u, v \in T_z\Omega$ .

Example: For the unit disc  $\Delta$  :  $B(z) = 2(1 - |z|^2)^2 \text{ Id}$ , where  $\text{Id}$  is the identity map of  $T_z\Omega$ .

Notice that the Ricci form  $\rho$  of  $\omega_{\text{Berg}}$  is given by:

$$\rho := i \partial \bar{\partial} \log \det B(z) .$$

The bounded domain  $\Omega$  is Kähler-Einstein if and only if  $\omega_{\text{Berg}}$  and  $\rho$  are proportional, i.e.  $\rho = e \omega_{\text{Berg}}$ , for  $e \in \mathbb{R}$ .

**Remark** : If the bounded domain  $\Omega \subset \mathbb{C}^n$  is *homogeneous* then Bergman metric  $\omega_{\text{Berg}}$  is Kähler-Einstein. I do not know any example of a non homogeneous domain with a Kähler-Einstein Bergman metric.

# The bisymplectomorphism group of a bounded domain $\Omega \subset \mathbb{C}^n$ .

A bisymplectomorphism  $f \in \mathcal{B}(\Omega)$  can be described by using the Bergman operator  $B(z)$ .

**Proposition 0.3** *Let  $\Omega$  be a bounded domain. Then a diffeomorphism  $f \in \text{Diff}(\Omega)$  is a bisymplectomorphism if and only if it satisfies:*

- $f^*\omega_0 = \omega_0,$
- $B(f(z)) \circ d f(z) = d f(z) \circ B(z) \quad (z \in \Omega).$

# Bounded symmetric domains of rank one $D_n$ .

Let  $D_n \subset \mathbb{C}^n$  be the open unit ball of the standard Hermitian space  $\mathbb{C}^n$ , with Hermitian scalar product

$$(z | t) = \sum_{j=1}^n z_j \bar{t}_j$$

and associated norm  $|z|$ . That is to say,

$$D_n := \{z \in \mathbb{C}^n : |z| < 1\}.$$

Here is the description of the bisymplectomorphisms.

**Theorem 0.4** *The bisymplectomorphisms  $f \in \mathcal{B}(D_n)$  are the maps defined by*

$$f(z) = \gamma(|z|^2) u(z) \quad (z \in D_n),$$

where  $u \in U(n)$  and  $\gamma$  is a smooth function  $\gamma : [0, 1) \rightarrow S^1 \simeq U(1)$ .

# Bounded symmetric domains of rank one $D_n$ .

Sketch of the Proof of Theorem 0.4 :

The Bergman operator  $B(z)$  is given by:

$$B(z)(w) := 2(1 - |z|^2)(w - z(w | z)).$$

In particular, **notice** that for fixed  $z \in D_n$  the operator  $B(z) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  has two eigenspaces. Namely,  $V_z := \mathbb{C}.z$  and  $V_z^\perp$ .

That is to say

$$\mathbb{C}^n = V_z \oplus V_z^\perp ,$$

where  $V_z$  and  $V_z^\perp$  are  $B(z)$ -invariant.

Then Proposition 0.3 implies that  $f$  must infinitesimally preserve such decomposition.

So if  $f \in \mathcal{B}(D_n)$  we get:

- $|f(z)| = |z|$ . Thus,  $df(0)$  is unitary, i.e.  $df(0) \in U(n)$ .
- $df(z)$  preserves the complex line  $l_z \subset T_z D_n$  spanned by  $z$ , i.e.  $l_z := \{w \in T_z D_n : w = \lambda z\}$ .
- Indeed,  $f$  takes complex lines through the origin into complex lines through the origin.

Notice that the complex lines through the origin are the *complex totally geodesic discs*  $\Delta$  of the symmetric domain, i.e. the complexifications of the flats.

Now we can restrict  $f$  to the discs  $\Delta \subset D_n$  to finish the proof.

$$\Delta^n := \underbrace{\Delta \times \cdots \times \Delta}_{n\text{-times}} \subset \mathbb{C}^n.$$

Here is the description of the bisymplectomorphisms of the polydisc.

**Theorem 0.5** *A map  $f : \Delta^n \rightarrow \Delta^n$  is a bisymplectomorphism if and only if there exist a permutation  $\sigma \in \mathfrak{S}_n$  and smooth functions  $u_j : [0, 1) \rightarrow S^1$  such that:*

$$f(z_1, \dots, z_r) = \sum_{j=1}^n u_j(|z_j|^2) z_j e_{\sigma(j)} \quad (z_j \in \Delta),$$

where  $(e_1, \dots, e_n)$  is the canonical basis of  $\mathbb{C}^n$ .

**The polydisc**

$$\Delta^n := \underbrace{\Delta \times \cdots \times \Delta}_{n\text{-times}} \subset \mathbb{C}^n.$$

Sketch of the Proof of Theorem 0.5 :

The Bergman operator  $B(z)$  is given by:

$$B(z_1, \dots, z_n) := \underbrace{B_\Delta(z_1) \times \cdots \times B_\Delta(z_1)}_{n\text{-times}}$$

where  $B_\Delta(z)$  is the Bergman operator of the unit disc  $\Delta$ .

- if  $f \in \mathcal{B}(\Delta^n)$  then Proposition 0.3 implies that (up to a permutation  $\sigma \in \mathfrak{S}_n$ )  $f$  splits.
  
- To finish the proof we can restrict  $f$  to each factor disc  $\Delta$ .

# Bounded symmetric domains $\Omega$ of higher rank.

The rank one case (and the polydisc) show that the description of  $\mathcal{B}(\Omega)$  depends upon a good algebraic description of the Bergman operator  $B(z)$ .

The theory of **Jordan Algebras** gives an algebraic description of the Bergman operator  $B(z)$  of all Bounded symmetric domains  $\Omega$ .

A principal role is played by the so called **Peirce simultaneous decomposition** relative to  $z \in \Omega$ . That is exactly the generalization of the decomposition  $\mathbb{C}^n = V_z \oplus V_z^\perp$  for the rank one case.

# Bounded symmetric domains from the point of View of Jordan Algebras.

There is a way, due to Max Koecher, to construct all the symmetric bounded domains  $\Omega \subset \mathbb{C}^n$  by starting with a **Hermitian Positive Jordan Triple System**  $(V, \{, , \})$  :

- $V = \mathbb{C}^n$  and  $\{, , \} : V^3 \rightarrow V$ ,
- $\{x, y, z\}$  is  $\mathbb{C}$ -bilinear in  $(x, z)$  and  $\mathbb{C}$ -anti-linear in  $y$ .
- satisfying the *Jordan identity* :

$$\begin{aligned} \{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} = \\ = \{\{x, y, u\}, v, w\} - \{u, \{v, x, y\}, w\}. \end{aligned}$$

- the sesquilinear form  $(x | y) := \text{trace} D(x, y)$  is positive, where  $D(x, y)(\cdot) := \{x, y, \cdot\}$ .

# Bounded symmetric domains from the point of View of Jordan Algebras.

- The **Spectral Norm**  $|z|$  of  $z \in V$  is defined as

$$|z|^2 := \frac{\|D(z, z)\|}{2}$$

where  $\|\cdot\|$  is the operator norm in  $V$  endowed with  $(\cdot|\cdot)$ .

- The bounded domain attached to the HPJTS  $(V, \{, , \})$  is given by:

$$\Omega := \{z \in V : |z| < 1\} .$$

That is to say,  $\Omega$  is the unit sphere w.r.t the Spectral Norm .

# Main Theorems.

Let  $\Omega \subset V = \mathbb{C}^n$  be an irreducible bounded symmetric domain attached to the HPJTS  $(V, \{, , \})$  of rank  $r$ .

Let us call **radial** a bisymplectomorphism  $f \in \mathcal{B}(\Omega)$  such

$$f(\Delta^r) = \Delta^r ,$$

for all polydiscs  $\Delta^r \subset \Omega$ .

**Theorem 0.6** *Any  $f \in \mathcal{B}(\Omega)$  is of the form*

$$f = u \circ R ,$$

*where  $R$  is a radial bisymplectomorphism and  $u = d f(0) \in K$  the isotropy group at  $0 \in \Omega$ .*

**Theorem 0.7** *Let  $R$  be a radial bisymplectomorphism. Then there exists a function  $h \in C^\infty[0, 1)$  such that*

$$R(z) = e^{ih(\lambda_1^2)} \lambda_1 e_1 + e^{ih(\lambda_2^2)} \lambda_2 e_2 + \cdots + e^{ih(\lambda_r^2)} \lambda_r e_r$$

*for all  $z \in M$ , where  $z = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_r e_r$  is the spectral decomposition of  $z \in M$ .*

## References

- [DL] Di Scala, A.J. and Loi, A., *Symplectic duality of symmetric spaces*, arXiv math.DG/0603141.
- [L] Loos, Ottmar, *Bounded symmetric domains and Jordan pairs*, Math. Lectures, Univ. of California, Irvine, 1977.
- [MD] McDuff, Dusa, The symplectic structure of Kähler manifolds of non-positive curvature, *J. Differ. Geom.* **28**, No.3, 467-475 (1988).
- [M] Mok, N., *Metric rigidity theorems on Hermitian locally symmetric spaces*, Series in Pure Mathematics, vol.6, World Scientific, [town?], 1989.
- [R] Roos, Guy, Jordan triple systems, pp. 425–534, in *J. Faraut, S. Kaneyuki, A. Korányi, Q.k. Lu, G. Roos, Analysis and geometry on complex homogeneous domains*, Progress in Mathematics, vol.185, Birkhäuser, Boston, 2000.
- [S] Schwarz, G.W., Smooth functions invariant under the action of a compact Lie group, *Topology.*, **14**, (1975).63–68.
- [W] Whitney, H., Differentiable even functions, *Duke Math. J.*, **10** (1943), 159–160.

Dipartimento di Matematica,  
Politecnico di Torino,  
Corso Duca degli Abruzzi 24, 10129 Torino, Italy.  
antonio.discal@polito.it