

Il gruppo dei bi-simplettomorfismi degli spazi simmetrici Hermitiani *

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The unit disc $\Delta \subset \mathbb{C}$.

The unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ has two well-known symplectic forms ω_0 and ω_{hyp} :

$$\omega_0 = \frac{i}{2} dz \wedge d\bar{z},$$
$$\omega_{hyp} = \frac{\omega_0}{(1 - |z|^2)^2}$$

Let $\mathcal{B}(\Delta)$ be the group of diffeomorphisms of Δ such that:

$$f^*(\omega_0) = \omega_0,$$
$$f^*(\omega_{hyp}) = \omega_{hyp}.$$

The group $\mathcal{B}(\Delta)$ is called *the group of bisymplectomorphisms of the disc Δ* .

Notice that the group $U(1) \subset \mathcal{B}(\Delta)$.

The unit disc $\Delta \subset \mathbb{C}$.

The following theorem characterizes the elements of $\mathcal{B}(\Delta)$.

Theorem 0.1 *The elements $f \in \mathcal{B}(\Delta)$ are the maps defined by*

$$f(z) = u(|z|^2) z \quad (z \in \Delta),$$

where u is a smooth function $u : [0, 1) \rightarrow S^1 \simeq U(1)$.

In other words, the restriction of a bisymplectomorphism $f \in \mathcal{B}(\Delta)$ to a circle of radius r ($0 < r < 1$) is the rotation $u(r^2)$.

Notice that if $f \in \mathcal{B}(\Delta)$ then $f(0) = 0$.

The unit disc $\Delta \in \mathbb{C}$.

Sketch of the Proof of Theorem 0.1 :

- Since f preserves both symplectic forms then f preserves the quotient $\frac{\omega_0}{\omega_{hyp}} = (1 - |z|^2)^2$. Thus,

$$|f(z)| = |z|$$

for $z \in \Delta$.

- Conversely if $|f(z)| = |z|$ for $z \in \Delta$ and f preserves ω_0 then $f \in \mathcal{B}(\Delta)$.
- It is not difficult to show that the maps $f(z) = u(|z|^2) z$ are bisymplectomorphisms.
- A simple computation shows that $f(z) = v(|z|)z$ for $z \in \Delta \setminus \{0\}$ and $v : (0, 1) \rightarrow U(1)$ smooth.
- Whitney's Theorem can be used to show that $v(|z|) = u(|z|^2)$ for a smooth u . \square

The bisymplectomorphism group of a bounded domain $\Omega \subset \mathbb{C}^n$.

A bounded domain $\Omega \subset \mathbb{C}^n$ has two well-known symplectic forms. Namely,

- The flat euclidean ω_0 :

$$\omega_0 = \frac{i}{2} dz \wedge d\bar{z},$$

- The Bergman form:

$$\omega_{\text{Berg}}(z) = \frac{i}{2} \partial \bar{\partial} \log K(z) ,$$

where $K(z) := \sum_{j=1}^{\infty} \phi_j(z) \overline{\phi_j(z)}$ and $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis in the space of holomorphic square integrable functions on Ω .

Example: For the unit disc Δ : $K(z) = \frac{1}{\pi}(1 - |z|^2)^{-2}$.

So, $2\omega_{\text{hyp}} = \omega_{\text{Berg}}$.

The bisymplectomorphism group of a bounded domain $\Omega \subset \mathbb{C}^n$.

Definition 0.2 A bisymplectomorphism of Ω is a diffeomorphism $f : \Omega \rightarrow \Omega$ which satisfies

$$f^* \omega_0 = \omega_0, \tag{1}$$

$$f^* \omega_{\text{Berg}} = \omega_{\text{Berg}}, \tag{2}$$

that is, which preserves both symplectic forms.

Notice that $f \in \mathcal{B}(\Omega)$ if and only if f preserves all linear combinations $a \omega_0 + b \omega_{\text{Berg}}$, $a, b \in \mathbb{R}$. Thus, f also preserves the so called "hyperbolic metric" ω_{hyp} .

The group $\mathcal{B}(\Omega)$ is called *the group of bisymplectomorphisms of the bounded domain Ω* .

The bisymplectomorphism group of a bounded domain $\Omega \subset \mathbb{C}^n$.

Since both forms ω_0 and ω_{Berg} are non degenerated there exists an operator $B(z)$, the Bergman operator, such that:

$$\omega_{\text{Berg}}(B(z)u, v) = \omega_0(u, v)$$

for $u, v \in T_z\Omega$.

Example: For the unit disc Δ : $B(z) = 2(1 - |z|^2)^2 \text{ Id}$, where Id is the identity map of $T_z\Omega$.

Notice that the Ricci form ρ of ω_{Berg} is given by:

$$\rho := i \partial \bar{\partial} \log \det B(z) .$$

The bounded domain Ω is Kähler-Einstein if and only if ω_{Berg} and ρ are proportional, i.e. $\rho = e \omega_{\text{Berg}}$, for $e \in \mathbb{R}$.

Remark : If the bounded domain $\Omega \subset \mathbb{C}^n$ is *homogeneous* then Bergman metric ω_{Berg} is Kähler-Einstein. I do not know any example of a non homogeneous domain with a Kähler-Einstein Bergman metric.

The bisymplectomorphism group of a bounded domain $\Omega \subset \mathbb{C}^n$.

A bisymplectomorphism $f \in \mathcal{B}(\Omega)$ can be described by using the Bergman operator $B(z)$.

Proposition 0.3 *Let Ω be a bounded domain. Then a diffeomorphism $f \in \text{Diff}(\Omega)$ is a bisymplectomorphism if and only if it satisfies:*

- $f^*\omega_0 = \omega_0,$
- $B(f(z)) \circ d f(z) = d f(z) \circ B(z) \quad (z \in \Omega).$

Bounded symmetric domains of rank one D_n .

Let $D_n \subset \mathbb{C}^n$ be the open unit ball of the standard Hermitian space \mathbb{C}^n , with Hermitian scalar product

$$(z | t) = \sum_{j=1}^n z_j \bar{t}_j$$

and associated norm $|z|$. That is to say,

$$D_n := \{z \in \mathbb{C}^n : |z| < 1\}.$$

Here is the description of the bisymplectomorphisms.

Theorem 0.4 *The bisymplectomorphisms $f \in \mathcal{B}(D_n)$ are the maps defined by*

$$f(z) = \gamma(|z|^2) u(z) \quad (z \in D_n),$$

where $u \in U(n)$ and γ is a smooth function $\gamma : [0, 1) \rightarrow S^1 \simeq U(1)$.

Bounded symmetric domains of rank one D_n .

Sketch of the Proof of Theorem 0.4 :

The Bergman operator $B(z)$ is given by:

$$B(z)(w) := 2(1 - |z|^2)(w - z(w | z)).$$

In particular, **notice** that for fixed $z \in D_n$ the operator $B(z) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ has two eigenspaces. Namely, $V_z := \mathbb{C}.z$ and V_z^\perp .

That is to say

$$\mathbb{C}^n = V_z \oplus V_z^\perp ,$$

where V_z and V_z^\perp are $B(z)$ -invariant.

Then Proposition 0.3 implies that f must infinitesimally preserve such decomposition.

So if $f \in \mathcal{B}(D_n)$ we get:

- $|f(z)| = |z|$. Thus, $df(0)$ is unitary, i.e. $df(0) \in U(n)$.
- $df(z)$ preserves the complex line $l_z \subset T_z D_n$ spanned by z , i.e. $l_z := \{w \in T_z D_n : w = \lambda z\}$.
- Indeed, f takes complex lines through the origin into complex lines through the origin.

Notice that the complex lines through the origin are the *complex totally geodesic discs* Δ of the symmetric domain, i.e. the complexifications of the flats.

Now we can restrict f to the discs $\Delta \subset D_n$ to finish the proof.

$$\Delta^n := \underbrace{\Delta \times \cdots \times \Delta}_{n\text{-times}} \subset \mathbb{C}^n.$$

Here is the description of the bisymplectomorphisms of the polydisc.

Theorem 0.5 *A map $f : \Delta^n \rightarrow \Delta^n$ is a bisymplectomorphism if and only if there exist a permutation $\sigma \in \mathfrak{S}_n$ and smooth functions $u_j : [0, 1) \rightarrow S^1$ such that:*

$$f(z_1, \dots, z_r) = \sum_{j=1}^n u_j(|z_j|^2) z_j e_{\sigma(j)} \quad (z_j \in \Delta),$$

where (e_1, \dots, e_n) is the canonical basis of \mathbb{C}^n .

The polydisc

$$\Delta^n := \underbrace{\Delta \times \cdots \times \Delta}_{n\text{-times}} \subset \mathbb{C}^n.$$

Sketch of the Proof of Theorem 0.5 :

The Bergman operator $B(z)$ is given by:

$$B(z_1, \dots, z_n) := \underbrace{B_\Delta(z_1) \times \cdots \times B_\Delta(z_1)}_{n\text{-times}}$$

where $B_\Delta(z)$ is the Bergman operator of the unit disc Δ .

- if $f \in \mathcal{B}(\Delta^n)$ then Proposition 0.3 implies that (up to a permutation $\sigma \in \mathfrak{S}_n$) f splits.

- To finish the proof we can restrict f to each factor disc Δ .

Bounded symmetric domains Ω of higher rank.

The rank one case (and the polydisc) show that the description of $\mathcal{B}(\Omega)$ depends upon a good algebraic description of the Bergman operator $B(z)$.

The theory of **Jordan Algebras** gives an algebraic description of the Bergman operator $B(z)$ of all Bounded symmetric domains Ω .

A principal role is played by the so called **Peirce simultaneous decomposition** relative to $z \in \Omega$. That is exactly the generalization of the decomposition $\mathbb{C}^n = V_z \oplus V_z^\perp$ for the rank one case.

Bounded symmetric domains from the point of View of Jordan Algebras.

There is a way, due to Max Koecher, to construct all the symmetric bounded domains $\Omega \subset \mathbb{C}^n$ by starting with a **Hermitian Positive Jordan Triple System** $(V, \{, , \})$:

- $V = \mathbb{C}^n$ and $\{, , \} : V^3 \rightarrow V$,
- $\{x, y, z\}$ is \mathbb{C} -bilinear in (x, z) and \mathbb{C} -anti-linear in y .
- satisfying the *Jordan identity* :

$$\begin{aligned} \{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} = \\ = \{\{x, y, u\}, v, w\} - \{u, \{v, x, y\}, w\}. \end{aligned}$$

- the sesquilinear form $(x | y) := \text{trace} D(x, y)$ is positive, where $D(x, y)(\cdot) := \{x, y, \cdot\}$.

Bounded symmetric domains from the point of View of Jordan Algebras.

- The **Spectral Norm** $|z|$ of $z \in V$ is defined as

$$|z|^2 := \frac{\|D(z, z)\|}{2}$$

where $\|\cdot\|$ is the operator norm in V endowed with $(\cdot|\cdot)$.

- The bounded domain attached to the HPJTS $(V, \{, , \})$ is given by:

$$\Omega := \{z \in V : |z| < 1\} .$$

That is to say, Ω is the unit sphere w.r.t the Spectral Norm .

Main Theorems.

Let $\Omega \subset V = \mathbb{C}^n$ be an irreducible bounded symmetric domain attached to the HPJTS $(V, \{, , \})$ of rank r .

Let us call **radial** a bisymplectomorphism $f \in \mathcal{B}(\Omega)$ such

$$f(\Delta^r) = \Delta^r ,$$

for all polydiscs $\Delta^r \subset \Omega$.

Theorem 0.6 *Any $f \in \mathcal{B}(\Omega)$ is of the form*

$$f = u \circ R ,$$

where R is a radial bisymplectomorphism and $u = \mathrm{d} f(0) \in K$ the isotropy group at $0 \in \Omega$.

Theorem 0.7 *Let R be a radial bisymplectomorphism. Then there exists a function $h \in C^\infty[0, 1)$ such that*

$$R(z) = e^{ih(\lambda_1^2)} \lambda_1 e_1 + e^{ih(\lambda_2^2)} \lambda_2 e_2 + \cdots + e^{ih(\lambda_r^2)} \lambda_r e_r$$

for all $z \in M$, where $z = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_r e_r$ is the spectral decomposition of $z \in M$.

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