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An introduction to the mathematical framework of the loop quantization of gauge theories and gravity

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Introduction

These notes are the collection of a series of seminars kept by the author at the University of Turin−Department of Mathematics in the March of 2002 within the course "Metodi geometrici ed omologici della fisica-matematica" kept by Lorenzo Fatibene and Marcella Palese.

The aim of them is to present some of the most important results obtained in the last decade in the mathematical formulation of the so-called "loop quantization" of gauge theories.

This is a canonical and non-perturbative procedure to quantize the gauge theories. The adjective "loop" is referred to the fact that such theories are formulated in terms of the well known Wilson's loop functions (the traces of the holonomies) instead of connections on a principal fiber bundle.

The great advantage to assume the Wilson functions as the classical configuration variables relies in the fact that they are gauge-invariant, hence the Gauss constraint is intrinsically solved already at a classical level, so that one doesn't need to resort to technical tricks as ghosts or gauge fixings and eliminates all the problems related to the Gribov ambiguities.

Furthermore, with techniques imported from lattice gauge theory and extended to the continuum, one can construct a natural compactification of the classical configuration space (the space of connections modulo gauge transformations) on which a quantum theory, at least at a kinematical level, can be implemented in a rigorous way.

Ashtekar's discovery of the "new variables" [3], in terms of which general relativity becomes a gauge theory with constraint imposed by the diffeomorphism invariance, imposed the problem to incorporate in the algorithm of the loop quantization even this constraint in order to finally get a consistent quantum formulation of gravity.

The concepts and results underlying the loop quantization belong to many different fields of physics and mathematics, for the sake of clarity I decided to describe them in separated chapters, each necessary to develop the next.

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Chapter 1

Principal fiber bundles, gauge theories and gravity

Initial assumption: the **manifolds** contemplated in this chapter will be assumed to be **ordinary**, i.e. smooth, connected, paracompact and finite dimensional, the **maps** between them will be assumed to be **smooth**, i.e. $\mathcal{C}^\infty$.

The aim of this chapter is to illustrate the mathematical framework of the gauge theories. The natural mathematical setting of these theories is that of principal fiber bundles, so I begin by introducing such objects.

1.1 Principal fiber bundles (PFB)

The classical reference for the proofs of the theorems cited in this chapter is [26]; more modern books, which also contain explicit applications to gauge theories, are [23] or [42].

Def. 1.1.1 *A principal fiber bundle is denoted usually as $P \equiv P(M, G)$ or $\pi : P \to M$ and it is the collection of the following objects satisfying the indicated properties:*

- *a manifold P, called* **total space***;*
- *a manifold M, called* **base space***;*
- *a Lie group G, called* **structure group** *from mathematicians and* **gauge group** *from physicists;*
- *a* **free** *right action of G on P*

$$R : P \times G \to P, \qquad (p, g) \mapsto R(p, g) \equiv p.g$$

which is **transitive on the fibers** *of P;*

- *a* **surjective submersion** $\pi : P \to M$ *called* **projection** *with the property of* **local triviality**, *i.e. there exists an open covering $\{U_\alpha\}$ of M and diffeomorphisms*

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$$

such that

$$\psi_\alpha(p) = (\pi(p), \phi_\alpha(p)) \qquad \forall p \in \pi^{-1}(U_\alpha)$$

where

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to G$$

is a G-equivariant map, i.e. : $\phi_\alpha(p.g) = \phi_\alpha(p)g$, $\forall g \in G$.

Remarks.

1. The collection $\{(U_\alpha, \psi_\alpha)\}$ is said to be a **locally trivializing collection** for $P(M,G)$ and every couple (U_α, ψ_α), is said a **local trivialization** or, in the language of gauge theories, a **choice of gauge**;

2. Fixed a point $x \in M$, the **fiber over** x is the inverse-image of x via π. The usual symbols used to denote the fiber over x are $\pi^{-1}(x)$ and P_x, both of them will be used in the sequel;

3. $P(M,G)$ is said to be **trivial** when $P \equiv M \times G$.

From the requires included in the definition of PFB it follows that the p-reduced of the right action, i.e. the map

$$R_p : G \to P, \qquad g \mapsto R_p(g) := p.g$$

is a diffeomorphism between G and the fiber over $\pi(p)$, for every p, hence **all the fibers of P are set-theoretically isomorphic to G.**

Analogously the g-reduced of R, i.e. the map

$$R_g : P \to P, \qquad p \mapsto R_g(p) := p.g$$

is a diffeomorphism of P into itself for every $g \in G$.

Furthermore, from the fact that R is everywhere free and transitive on the fibers it follows that, fixed an arbitrary point p_0 in an assigned fiber, for every other point p belonging to the same fiber there exists one and only one $g \in G$ such that $p = p_0.g$.

Observe now that if $\{(U_\alpha, \psi_\alpha)\}$ is a locally trivializing collection for $P(M,G)$ and if the point $p \in P$ belongs to $\pi^{-1}(U_\alpha \cap U_\beta)$ then:

$$\phi_\beta(p.g)(\phi_\alpha(p.g))^{-1} = \phi_\beta(p).gg^{-1}.(\phi_\alpha(p))^{-1} = \phi_\beta(p)(\phi_\alpha(p))^{-1} \quad \forall g \in G,$$

this shows that $\phi_\beta(p)(\phi_\alpha(p))^{-1}$ doesn't depend on p but only on $\pi(p)$, hence it is well posed the next definition.

Def. 1.1.2 *The* **transition functions** *of $P(M,G)$ are the maps:*

$$\begin{array}{rccl} g_{\alpha\beta}: & U_\alpha \cap U_\beta & \to & G \\ & \pi(p) & \mapsto & g_{\beta\alpha}(\pi(p)) := \phi_\alpha(p)(\phi_\beta(p))^{-1} \end{array}$$

$\forall p \in \pi^{-1}(U_\alpha \cap U_\beta)$.

A map $\sigma : M \to P$ is called a **section of** $P(M,G)$ if $\pi(\sigma(x)) = x$ for every $x \in M$. It can be proved that a PFB doesn't possess globally defined sections, unless it is trivial, but it always possess **local sections**, the maps defined below.

Def. 1.1.3 *The local sections of $P(M,G)$ w.r.t. the locally trivializing collection $\{(U_\alpha, \psi_\alpha)\}$ are the following maps:*

$$\begin{array}{rccl} \sigma_\alpha: & U_\alpha & \to & \pi^{-1}(U_\alpha) \\ & x & \mapsto & \sigma_\alpha(x) := \psi_\alpha^{-1}(x,e) \end{array}$$

where e is the unit of G.

It is evident that **the choice of a local section is equivalent to a choice of a particular gauge**.

The concepts of transition functions and local sections will be used in section 1.3 to write down an important transformation rule.

1.2 Gauge transformations in a PFB

The transformations more connatural with the structure of a PFB are the gauge transformations, to define them it is necessary to start introducing the concept of **automorphism of a PFB**.

Def. 1.2.1 *An automorphism of $P(M,G)$ is a G-equivariant diffeomorphism $\Phi : P \to P$, i.e. $\Phi(p.g) = \Phi(p).g \quad \forall\, p \in P, \forall g \in G$.*

The automorphisms of a PFB form a group w.r.t. functional composition, this group is indicated by $\mathrm{Aut}(P)$.

Every $\Phi \in \mathrm{Aut}(P)$ induces in a unique way a diffeomorphism $\Psi : M \to M$ of the base space by:

$$\Psi(\pi(p)) := \pi(\Phi(p)) \qquad \forall p \in P \tag{1.1}$$

well defined because π is a surjection.

If $\mathrm{Diff}(M)$ is the group of the diffeomorphisms of M, then the map:

$$\begin{array}{rccc} \flat : & \mathrm{Aut}(P) & \longrightarrow & \mathrm{Diff}(M) \\ & \Phi & \mapsto & \flat(\Phi) = \Psi \end{array}$$

is a group homomorphism.

Def. 1.2.2 *The* **gauge transformations** *of* $P(M, G)$ *are the automorphisms* $\Phi \in$ *Aut(P) inducing the identity diffeomorphism on* M. *Such automorphisms are also said to be* **strong** *or* **vertical**.

Directly from the definition it follows that the set of the gauge transformations of $P(M, G)$ is precisely $Ker(\flat)$, hence it is a normal subgroup of $\mathrm{Aut}(P)$. Such a group is usually denoted by $\mathrm{Gau}(P)$ or by $\mathcal{G}$ and for trivial bundles it can be shown to agree with $\mathcal{C}^\infty(M, G)$.

Geometrically *the gauge transformations* are often described by saying that they *leave untouched the point on the base space and move the elemnts of the fiber over that point.*

1.3 Connections in a PFB

The tangent bundle $T(P)$ of the total space P of a principal fiber bundle always possess, in a natural way, a sub-bundle indicated with $\mathrm{Ver}(P)$ and called **vertical sub-bundle**, whose fiber in the generic point $p \in P$ is:

$$V_pP := T_p(P_{\pi(p)})$$

i.e. the subspace of T_pP given by the tangent vectors to the elements of the fiber to which p belongs.

However a PFB doesn't possess in a natural way a sub-bundle which is supplementary to the vertical one. The presence of such a sub-bundle would be very useful, because in this situation $T(P)$ would be decomposed into a direct sum of sub-bundles.

A principal connection is exactly the instrument which generates this decomposition, as one can immediately see from its formal definition.

Def. 1.3.1 *A principal connection* Γ *on* $P(M,G)$ *is a smooth* G*-equivariant assignment of a sub-bundle* $Hor(P)$ *of* $T(P)$ *supplementary to* $Ver(P)$*, i.e. this assignment satisfies:*

$$1)\ T(P) \simeq Ver(P) \oplus Hor(P);$$

$$2)\ H_{p.g}P = (R_g)_* \, H_pP \qquad \forall p \in P,\ \forall g \in G.$$

where H_pP *is the fiber of* $Hor(P)$ *to which* p *belongs and* $(R_g)_*$ *is the push-forward of* R_g*.*

$Hor(P)$ *is called the* **horizontal sub-bundle** *of* $P(M,G)$*.*

The require 2) is introduced to have compatibility between the vertical vs. orthogonal decomposition of $T(P)$ and the right action of G on P.

Under the topological assumption of paracompactness for the manifolds involved (as assumed since the begin) one can prove that **every PFB admits a principal connection**.

Since the connections considered in the sequel will always be principal, I shall omit this adjective.

Thanks to the presence of a connection one can define the **vertical and horizontal vector fields and 1-forms on** P:

Def. 1.3.2 *Let* $P(M,G)$ *be a PFB with a fixed connection* Γ*. A vector field* X *on* P *is said to be vertical (resp. horizontal) if* $X_p \in V_pP$ *(resp.* $X_p \in H_pP$*),* $\forall p \in P$*.*

Analogously a 1-form on P *is said vertical (resp. horizontal) if it takes identically zero values when it is calculated on the horizontal (resp. vertical) vector fields of* P*.*

From the splitting induced by a fixed principal connection, it follows that every vector field X on P can be decomposed in a unique way as the orthogonal sum of its **vertical component** X^v and its **horizontal component** X^h.

The concept of connection as defined above is important to understand the geometrical consequences induced on a PFB by its presence, but, as will be discussed in the last section of this chapter, in the applications to gauge theories it is more useful to work with a 1-form closely related to the connection and for this reason called **connection 1-form**.

To introduce the connections 1-form it is necessary to define the **fundamental vector fields** on P.

Def. 1.3.3 *For every vector field Y on G, the vector field $\tilde{Y}$ on P defined by:*

$$\tilde{Y}_p := (R_p)_* Y$$

is called the fundamental vector field associated to Y, which, by converse, is called the **generator** *of $\tilde{Y}$.*

The fundamental vector fields are easily seen to be vertical vector fields on P, moreover the next theorem holds.

Theorem 1.3.1 *The following map between $\mathfrak{g}$, the Lie algebra of G, and the space of the vertical vector field on P,*

$$\begin{array}{ccc} \mathfrak{g} & \to & Ver(P) \\ Y & \mapsto & \tilde{Y} \end{array}$$

is a bijection.

Indicated with $\Lambda(P;\mathfrak{g})$ the set of all the $\mathfrak{g}$-valued 1-forms on P the definition of a connection 1-form can be stated as below.

Def. 1.3.4 *A $\mathfrak{g}$-valued 1-form $A \in \Lambda(P;\mathfrak{g})$ which has these properties:*

$$a)\ A(\tilde{Y}) = Y \qquad \forall Y \in \mathfrak{g}\,;$$

$$b)\ (R_g)^* A = Ad_{g^{-1}} A \qquad \forall g \in G\,.$$

is called a connection 1-form.

The request $a)$ means that A reproduces the generators of the fundamental vector fields, while $b)$ express the equivariance of A.

The correlation between the connections on a PFB and the connection 1-forms is contained in the following theorem.

Theorem 1.3.2 *There is a bijection between the set of the connections on P and the set of the connection 1-forms on P.*

Thanks to the bijection expressed in the above theorem, it is common to call simply connections the connection 1-forms, in the sequel this convention will be adopted.

It can be proved that the set of all connections A have the structure of an affine manifold. The typical symbol used to indicate it is $\mathcal{A}$.

The local expressions of a connection A are obtained by taking the pull-back of A w.r.t. the local sections σ_α of $P(M,G)$. This operation gives rise to $\mathfrak{g}$-valued 1-forms locally defined on M:

$$A_\alpha := {\sigma_\alpha}^* A \ \in \Lambda(U_\alpha;\mathfrak{g})$$

for every choice of gauge (U_α, ψ_α), or of a local section σ_α.

To understand the behavior of A_α under a change of gauge it is necessary to introduce a natural $\mathfrak{g}$-valued 1-form on the gauge group G:

Def. 1.3.5 *The* **Maurer-Cartan 1-form** $\Theta \in \Lambda(G;\mathfrak{g})$ *is defined by:*

$$\Theta(X) := (L_{g^{-1}})_*(X) \qquad g \in G,\ X \in T_gG$$

where $L_{g^{-1}}$ is the left translation in G by the element g^{-1}.

By taking the pull-back of Θ w.r.t. the transition functions $g_{\alpha\beta}$ of $P(M,G)$ one gets $\mathfrak{g}$-valued 1-forms locally defined on M, precisely:

$$\Theta_{\alpha\beta} := {g_{\alpha\beta}}^*\Theta \ \in \Lambda(U_\alpha \cap U_\beta;\mathfrak{g})$$

for every overlapping couple of local trivialization (U_α, ψ_α), (U_β, ψ_β).

The next theorem is one of the most important in the applications of the theory of connections to gauge theories.

Theorem 1.3.3 *The transformation law of the local 1-forms A_α on the intersection $U_\alpha \cap U_\beta$ is:*

$$A_\beta = Ad_{g_{\alpha\beta}^{-1}} A_\alpha + \Theta_{\alpha\beta}. \tag{1.2}$$

By converse, if the family $\{A_\alpha\} \subset \Lambda(U_\alpha;\mathfrak{g})$ satisfies(1.2) then it exists one and only one connection A which has A_α as local expressions, i.e.

$$A_\alpha = {\sigma_\alpha}^*A.$$

It is worth noting that (1.2) is an **affine transformation**, so that A_α doesn't define a tensorial quantity.

The (left) action of the group of the gauge transformations $\mathcal{G}$ of a fixed PFB on the affine manifold of the connections $\mathcal{A}$ (on the same PFB) is defined by:

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{A} & \to & \mathcal{A} \\ (\Phi, A) & \mapsto & \Phi^*A := A' \end{array}$$

where: $\Phi^*A(X) := A(\Phi_*X)$, for every vector field X on P.

Def. 1.3.6 *A' is called the* **gauge-transformed connection** *of the connection A. Moreover A and A' are said to be* **gauge-equivalent connections**.

The gauge-equivalence of connections is really an equivalence relation in $\mathcal{A}$ and the space of gauge-equivalent connections is indicated by $\mathcal{A}/\mathcal{G}$.

To conclude the panoramic on the theory of connections on a PFB it is worth introducing the concept of curvature of a connection.

Def. 1.3.7 *Let*

$$\begin{array}{rccl} h: & T(P) & \longrightarrow & Hor(P) \\ & X & \mapsto & h(X) := X^h \end{array}$$

be the projector operator on the subspace of horizontal vector fields on P, then the **covariant exterior derivative** *of a p-form ω is the $(p+1)$-form given by $D\omega := d\omega \circ h$, i.e.*

$$D\omega(X_1, \dots, X_{p+1}) := d\omega(X_1^h, \dots, X_{p+1}^h),$$

being d the external differential and $X_1, \dots, X_{p+1}$ any set of vector fields on P.

In particular, the covariant exterior derivative of a connection A is the 2-form $F \in \Lambda^2(P, \mathfrak{g})$, called **curvature of the connection** *A.*

The important relation between a connection and its curvature is the contained in the **Maurer-Cartan structural relation**:

Theorem 1.3.4 *The curvature F of a connection A on a PFB satisfies the following equation:*

$$F(X, Y) = dA(X, Y) + [A(X), A(Y)]$$

for any couple of vector fields X, Y on P. The bracket is obviously taken in $\mathfrak{g}$, the Lie algebra of G.

1.4 Gauge theories

The PFB provide a natural mathematical setting for gauge theories, the interested reader can find a wide discussion of this in [30].

A gauge theory can be defined as a field theory whose configuration space is $\mathcal{A}/\mathcal{G}$, i.e. whose states are parameterized by gauge-equivalence classes of connections on a principal fiber bundle.

The physical meaning of the objects which compose a PFB is the following:

- M can represent a Cauchy hypersurface embedded in the space-time of the theory or the space time itself; the first choice corresponds to the **canonical formulation**, while the second choice corresponds to the **Feynman formulation** of the gauge theory;
- G is the group of the **internal symmetries** of the theory;
- P is a super-imposed structure, an auxiliary space containing the fibers over the points of M (copies of G).

The prototype of all gauge theories is the Maxwell theory of electromagnetism: it is well known that the classical electromagnetic phenomena can be described in terms of the so called **tetravector potential** A_μ, $\mu = 0, \ldots, 3$, which, geometrically speaking, defines a connection on a trivial principal fiber bundle with gauge group $U(1)$ over the Minkowski space, i.e. $\mathbb{R}^4$ endowed with the metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. This connection is called **gauge potential**.

The **field strength** of the Maxwell field is then encoded in the curvature $F = F_{\mu\nu} dx^\mu dx^\nu$ of the connection A_μ, with $F_{\mu\nu} = (DA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that the term containing the bracket is zero since $\mathfrak{u}(1)$, the Lie algebra of $U(1)$, is Abelian.

The Maxwell equations can be written in the so-called covariant (or geometric) form, i.e. :

$$\begin{cases} dF = 0 & \text{Bianchi identity} \\ \star(d \star F) = J & \end{cases}$$

where J is the current, $\star F$ is called the **dual** of F and $\star$ is the **Hodge star operator**.

If M is a n-dimensional oriented manifold, the Hodge star operator is the unique linear operator $\star : \Lambda^p(M) \to \Lambda^{n-p}(M)$ such that $\omega \wedge \star\tilde{\omega} =< \omega, \tilde{\omega} > vol$, for all $\omega, \tilde{\omega} \in \Lambda^p(M)$, where $< \omega, \tilde{\omega} >$ is the inner product of the two p-forms and $vol := \sqrt{|det(g_{\mu\nu})|} dx^1 \wedge \ldots \wedge dx^n$, is the volume form (well defined for any oriented n-dimensional manifold M with metric g).

The crucial fact to stress is now that if A_μ satisfies Maxwell equations then even any other potential which differs from A_μ by a gauge transformation satisfies the same equations.

Hence the physically distinct electromagnetic configurations are described by gauge-equivalence classes of connections on a PFB with structure group $U(1)$, this makes the electromagnetism an Abelian gauge theory.

In 1954 Yang and Mills constructed non-Abelian gauge theories by replacing $U(1)$ with the compact semisimple non-Abelian Lie groups $U(N)$

and $SU(N)$, $N > 1$. Analogously to the electromagnetic case, the gauge potentials of the theory are connections, whose local expressions have components given by $A_\mu : U \subset M \to \mathfrak{g}$, $\mu = 0, \dots, 3$ (the local index α has been suppressed from A and U to get a clearer notation).

The fields equations which generalize the Maxwell equations to the non-Abelian case are **the Yang-Mills equations**:

$$\begin{cases} DF = 0 \\ \star(D \star F) = J. \end{cases}$$

The most important difference between the Abelian and non-Abelian case ([7], [8]) is that the curvature F of A has a non-linear dependence on the gauge potential itself, in fact the Maurer-Cartan structural equation in local coordinates (obtained by the commutator of the covariant derivative $D_\mu = \partial_\mu + A_\mu$) is:

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

where the commutator $[A_\mu, A_\nu]$ is taken in $\mathfrak{g}$ and so it doesn't vanish because of the non-Abelian nature of the gauge group.

This extra term in the curvature has a dramatic consequence on the Yang-Mills equations: due to this term these equations, unlike the Maxwell equations, are non-linear. By a physical point of view this non-linearity corresponds to the presence of *self-interactions in the physics of non-Abelian gauge theories*. This fact is, for example, responsible of the fact that photons (the quanta of the electromagnetic field) doesn't carry electric charge (the sources of the electromagnetic field), while the gluons (the quanta of the strong nuclear field), carries color (the sources of this field).

Note however that there are quantum-like experimental evidences, like the Aharonov-Bohm effect ([9] pages 130-140), which show that there can be physically observable effects of the gauge fields even where the curvature of the connection is zero, thus the interpretation of the curvature of a connection as the strength of the gauge field is physically consistent only locally, it can't represent the global configuration of the gauge field. This is a consequence of the fact that the curvature of a connections is not a gauge-invariant quantity.

Finally I'd like to remember that the gauge theories are exactly the theories contemplated in the **standard model** of the nuclear and electromagnetic interactions.

In this model the particles interact by exchanging quanta of the gauge fields representing the force which makes them interact.

Precisely:

- the *quantum chromodynamics* (QCD), which describes the strong nuclear interactions, is a quantized Yang-Mills theory with gauge group

$SU(3)$;

- the electromagnetic and the weak interactions are unified in the so called *electro-weak theory*, a quantized Yang-Mills theory with gauge group $SU(2) \times U(1)$. The decoupling between the weak interactions and the electromagnetic ones is described by a mechanism called *spontaneous symmetry breaking*.

The only other force in nature, the gravity, is described by Einstein's general relativity, to which is dedicated the following section.

1.5 Ashtekar's formulation of general relativity

Einstein's general relativity is the physical theory which describes how the distribution of matter and energy curves the geometry of the spacetime in which it is immersed. The way it happens is expressed by the Einstein equations.

This equations relate the **stress-energy tensor**, a symmetric $(0, 2)$ tensor ($T_{\mu\nu}$ in local coordinates) which express the flow of energy and momentum through a given point of spacetime, with the curvature of the **Levi-Civita connection** ∇ associated to the metric $g_{\mu\nu}$ of the spacetime manifold.

This curvature is expressed by means of the **Riemann tensor**, defined by: $R(X, Y)Z := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$, or, in local coordinates:

$$R^{\mu}_{\nu\lambda\gamma} = \partial_\nu \Gamma^{\mu}_{\lambda\gamma} - \partial_\lambda \Gamma^{\mu}_{\nu\gamma} + \Gamma^{\sigma}_{\lambda\gamma}\Gamma^{\mu}_{\nu\sigma} - \Gamma^{\sigma}_{\nu\gamma}\Gamma^{\mu}_{\lambda\sigma}$$

where the Γ's are the Christoffel symbols, related to the the partial derivatives of the coordinate of the metric in this way:

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\nu g_{\nu\lambda}).$$

The trace of this tensor gives rise to the **Ricci tensor**: $R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu}$ and the contraction of the Ricci tensor gives the **scalar curvature** $R = R^{\mu}{}_{\mu}$.

This objects appear, with the metric itself, in **the Einstein equations**:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

in units where Newton's gravitational constant is fixed to be 1.

$G_{\mu\nu}$ are the components of a symmetric $(0, 2)$ tensor named **Einstein's tensor**.

Due to the symmetries of the Ricci tensor, **the Einstein equations are 10 second order hyperbolic non-linear equations in the components of the metric tensor** for every 4-dimensional spacetime.

If one imposes the Cauchy problem on these equations suddenly understand that not all of them are evolutory equations, in fact 4 equations are constraints and the remaining 6 equations are evolutory equations.

The reason why this happens is better understood if one consider the variational formulation of general relativity.

For the sake of simplicity, the next discussion of the actions for gravity will be focused only in the vacuum, i.e. $T_{\mu\nu} = 0$.

The first action for gravity is the Einstein-Hilbert action, i.e. a functional S on the space $Lor(M)$ of all Lorentzian metrics on a 4-D spacetime M given by:

$$S(g) := \int_M R\, vol$$

where vol is the volume form induced by g, which can be written, in local coordinates, as $vol = \sqrt{|det(g)|}dx^0 \wedge \ldots \wedge dx^3$.

The variation of S is minimized precisely when the Einstein vacuum equations hold.

The important thing to note is that this action is invariant under the action of the orientation preserving diffeomorphisms ϕ of M, i.e.:

$$\int_M (\phi^* R) \phi^* vol = \int_M R\, vol.$$

In general, the presence of such local symmetries implies that the Euler-Lagrange equations variationally deduced from the action (in this case the vacuum Einstein equations) are not independent and the theory, both in the Lagrangian and in the Hamiltonian formulation, is submitted to **constraints**.

The constraints which appear in the Einstein-Hilbert general relativity are very difficult to handle because they have a non-polynomial character and they are not closed under Poisson brackets.

The most important consequence of Ashtekar's formulation of general relativity [3] is the simplification of the these constraints, which become polynomial, closed under Poisson brackets and functionally simpler. Moreover, there exists a quantum description of gravity in which these constraints are (at least partially) solved, this is the celebrated **loop representation** due to Rovelli and Smolin [38].

Ashtekar's work is deeply related to **the Palatini formalism**, in which one consider a parallelizable oriented 4-D manifold M, i.e. it assumes that there exists a vector bundle isomorphism

$$e : \tau \equiv M \times \mathbb{R}^4 \to TM$$

inducing the identity on M. $\mathbb{R}^4$ here is called the **internal space** and capital letters $I, J, \ldots$ are used to denote its coordinates.

If $\{\xi_I\}_{I=0,\ldots,3}$ is the standard base of sections of τ then the corresponding base of vector fields on M is $\{e_I \equiv e \circ \xi_I\}_{I=0,\ldots,3}$ and e_I is locally expressed as: $e_I = e_I^\alpha \partial_\alpha$.

The Minkowski metric on each fiber defines on τ the so-called **internal metric** η.

In general the map e is called a **frame** and if the basis $\{e_I\}$ is orthonormal w.r.t. a given Lorentzian metric g on M, i.e. if $g(e_I, e_J) = \eta_{IJ}$, then the map e is called a **tetrad** or a **vierbein** for g.

Conversely, e defines a metric g on M by the formula above.

The inverse map $e^{-1} : TM \to M \times \mathbb{R}^4$ has local coordinates e_I^α satisfying $e_I^\alpha e_\alpha^J = \delta_I^J$ and is called a **cotetrad**.

The important thing to stress now is that if M is parallelizable then its principal frame bundle $\mathcal{R}M$ is also trivializable and every frame generates a trivialization by

$$\begin{array}{rccc} T: & M \times GL(4) & \longrightarrow & \mathcal{R}M \\ & (x, G) & \mapsto & T(x, G) := \{G_I^J e^I(x)\}. \end{array}$$

By considering in particular the sub-bundle $M \times SO(3,1)$ of $M \times GL(4)$ one can construct the (first order) Palatini action:

$$S(e, A) := \int_M e_I^\alpha e_J^\beta F_{\alpha\beta}^{IJ} \, vol(e)$$

where e is a vierbein, A is a principal connection on $M \times SO(3,1)$, F is its curvature and $vol(e)$ is the volume form defined by the Lorentzian metric g expressed as a function of e, i.e. $g_{\alpha\beta} = \eta_{IJ} e_\alpha^I e_\beta^J$.

Thus the Palatini action is a functional of a connection A and a frame e, and it can be shown that varying S, with respect to both A and e, the equation $\delta S = 0$ implies that the metric $g_{\alpha\beta}$ satisfies Einstein's vacuum equations.

The Palatini formulation of general relativity has the remarkable characteristic to encode this theory in the framework of gauge theories, the price to pay is that now there are both gauge and diffeomorphism constraints, due to the invariance of $S(e, A)$ under both gauge transformations and diffeomorphisms.

Even though the form of these constraints is much simpler than in the Einstein-Hilbert approach (since they have a polynomial character), these constraints are not closed under Poisson brackets and this creates many difficulties in the canonical quantization of the theory.

Roughly speaking, Ashtekar has discovered that the Palatini action is built by using too much degrees of freedom then strictly necessary, in fact only the so-called self-dual part of F contains the geometrical information which lead to the Einstein equations. The most important consequence of the substitution of the self-dual part of F in the Palatini action is that the functional expression of the constraints simplifies and they become closed under Poisson brackets.

The starting point of Ashtekar's work is the recognition that, on the 4-dimensional Minkowski space M, the linear endomorphism given by the Hodge star operator $* : \bigwedge^2 M \to \bigwedge^2 M$ defined on the antisymmetric $(0,2)$ tensors as:

$$*F_{IJ} = \frac{1}{2}\epsilon_{IJ}^{KL} F_{KL}$$

doesn't admit eigenvalues, but if one complexifies the theory by taking $TM \simeq M \times \mathbb{C}^4$, the Hodge star operator has eigenvalues $\pm i$ and the space $\bigwedge^2 \mathbb{C}^4$ decomposes into the direct sum of its **self-dual** and **antiself-dual** subspaces:

$$\bigwedge^2 \mathbb{C}^4 = \bigwedge^2 (\mathbb{C}^4)^+ \oplus \bigwedge^2 (\mathbb{C}^4)^-$$

which are the eigenspaces relatives to the eigenvalues $\pm i$.

The important thing to observe now is that there exists the isomorphism $\bigwedge^2 \mathbb{C}^4 \simeq \mathfrak{so}(3,1) \otimes \mathbb{C}$ and, thanks to the existence of the double cover $\rho : SL(2,\mathbb{C}) \to SO_0(3,1)$, the above splitting of $\bigwedge^2(\mathbb{C}^4)$ into self-dual and antiself-dual part corresponds to the splitting

$$\mathfrak{so}(3,1) \otimes \mathbb{C} \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}).$$

Since a Lorentz connection A on $M \times \mathbb{C}^4$ is just an $\mathfrak{so}(3,1) \otimes \mathbb{C}$-valued 1-form on M, the self-dual part of this connection, written usually as ${}^+A$, is a $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form on M.

Thus Ashtekar modifies Palatini's formalism by introducing:

- the vector bundle $\mathbb{C}\tau = M \times \mathbb{C}^4$;
- the complexified tangent bundle $\mathbb{C}TM = \coprod_{x \in M} \mathbb{C} \otimes T_x M$;
- complex frame fields, i.e. vector bundle isomorphisms $e : \mathbb{C}\tau \to \mathbb{C}TM$;

and then defines an action, the so-called **Ashtekar's self-dual action for gravity** simply by taking the complexified Palatini action written in terms of the self-dual connection ${}^{+}A$ and the complex vierbein e:

$$S(e, {}^{+}A) := \int_M e_I^\alpha e_J^\beta {}^{+}F_{\alpha\beta}^{IJ} vol(e)$$

quite miraculously, by varying S both w.r.t. e and ${}^{+}A$, one gets again the vacuum Einstein equations for the complex valued metric $g_{\alpha\beta} = \eta_{IJ} e_\alpha^I e_\beta^J$.

To obtain the usual (real) gravitation one has two following possibilities:

1. to impose reality conditions on the complex frame fields in terms of which the metric is expressed;

2. to start from an Euclidean self-dual action, defined by a volume form $vol(e)$ on $\mathbb{R}^4$ induced by the (real) Riemannian metric

$$g_{\alpha\beta} = \delta_{IJ} e_\alpha^I e_\beta^J$$

and $\mathfrak{su}(2)$-valued self-dual connections ${}^{+}A$. By using the fact that $\mathfrak{su}(2)$ is the compact real form of $\mathfrak{sl}(2, \mathbb{C})$, one obtains again the (real) Einstein's equations. The relation between the Euclidean formulation and the Lorentzian formulation is then obtained with a generalized Wick transform, called **coherent state transform**, constructed from Ashtekar and his collaborators in [6].

The Euclidean self-dual action is invariant both under orientation preserving diffeomorphisms and under $SU(2)-$gauge transformations and constitutes the most important example of constrained gauge field theory to which the program of loop quantization applies.

Chapter 2

Loops and loop groups

2.1 Loops, paths and graphs embedded in a manifold

In this chapter will be presented the various types of loop groups available in literature and will be analyzed the relations between them.

In the following $P(M, G)$ will denote a PFB in which the base manifold M is taken to be an ordinary manifold of dimension $dim(M) > 1$ equipped with a fixed real analytic structure. The choice of this structure is due to the fact that, at the time of writing, the most important results in the developments of the loop quantization necessitate the use of piecewise analytic loops in M.

It is worth remembering some terminology about paths and loops: a **path** in M is a *continuous and piecewise* $\mathcal{C}^1$ map of the form:

$$\gamma : [a, b] \to M$$

$[a, b] \subset \mathbb{R}$ is called **domain of parameterization** of γ.

A **closed path** in M, i.e. a path for which $\gamma_a = \gamma_b$, is commonly called **loop** in M.

γ is said to be **analytic** on $[a, b]$ if it is the restriction of an analytic map defined on an open set containing $[a, b]$.

γ^* will indicate the image of the path γ, that is the subset of M given by $\{\gamma_t \in M \mid t \in [a, b]\}$.

An **arc** of γ is any subset $\gamma_i^* \subset \gamma^*$ such that γ_i^* is the image of an analytic restriction $\gamma_i : [t_i, t_{i+1}] \to M$ of γ.

The **composition of two paths**, say $\gamma_1 : [a, b] \to M$ and $\gamma_2 : [c, d] \to M$, is subjected to two conditions, in fact it can be defined only when $b = c$ and $\gamma_1(b) = \gamma_2(c)$; if this is the case then the composed path is $\gamma_1\gamma_2 : [a, d] \to M$, defined by

$$\gamma_1\gamma_2(t) := \begin{cases} \gamma_1(t) & t \in [a,b] \\ \gamma_2(t) & t \in [c,d] \end{cases}$$

thus $\gamma_1\gamma_2$ is simply the path obtained travelling first the path written on the left and then the path written on his right. Observe that **this convention is opposed to that commonly used to define composition of functions!** For this reason the composition of functions will be indicated by the symbol $\circ$ and the composition of paths will be denoted simply by juxtaposing the paths in the order shown above.

The composition of many paths is defined in an analogous fashion.

The **inverse path** γ^{-1} is simply the path γ travelled in the opposite direction, that is

$$\gamma^{-1} : [a,b] \to M,\ \gamma^{-1}(t) := \gamma(a+b-t)\,.$$

The set of paths in M equipped with the composition law defined above is a **groupoid**, that is, roughly speaking, a semigroup with composition law not always defined. More rigorously a groupoid is defined to be a set Λ endowed with a binary operation (indicated with the juxtaposition of its elements) satisfying the following properties:

- for every element $\lambda \in \Lambda$ there exists an element $\lambda^{-1} \in \Lambda$, called its inverse, such that $r(\lambda) := \lambda\lambda^{-1}$ and $l(\lambda) := \lambda^{-1}\lambda$ exist and are the right and the left unit of λ, respectively;

- for every $\lambda, \eta \in \Lambda$, the product $\lambda\eta$ is defined if and only if $r(\eta) = l(\lambda)$;

- when defined, the product is associative.

If $\mathsf{U}(\Lambda)$ denotes the set of the units of Λ, i.e. the collection of all the elements of the form $\lambda\lambda^{-1}$, for some $\lambda \in \Lambda$, it is obvious that a groupoid Λ is a group if and only if $\mathsf{U}(\Lambda)$ is a singleton.

There is an important concept concerning the parameterization of a path: given two paths γ and η in M with domains $[a,b]$ and $[c,d]$, respectively, if there exists a *diffeomorphism* $\tau : [a,b] \to [c,d]$ such that

$$\begin{array}{ccc} M & = & M \\ \gamma\uparrow & & \uparrow\eta \\ [a,b] & \xrightarrow[\tau]{} & [c,d] \end{array}$$

i.e. $\gamma(t) = \eta(\tau(t))$, $\forall t \in [a,b]$, then the paths γ and η are said to be one the **reparameterization** of the other. It is obvious that two reparameterized paths have identical image, but the same point on the common image is reached for different values of the parameter t.

Since τ is a real diffeomorphism, it must be monotone and this leads to the following classification:

- if τ is a growing function then γ and η are said to have the *same orientation*;
- if τ is a decreasing function then γ and η are said to have *opposite orientation.*

It is easy to see that the relation "orientation-preserving reparameterization" is an equivalence relation in the set of all path in M, thus it is well posed the following definition.

Def. 2.1.1 *An* **oriented path in** M *is an equivalence class of paths in* M *w.r.t. the equivalence relation "orientation-preserving reparameterization".*

For oriented paths it is usual to choose as domain of parameterization the real closed interval $[0,1]$, this choice is possible by virtue of the diffeomorphism

$$\begin{array}{ccc} [a,b] & \longrightarrow & [0,1] \\ t & \mapsto & \frac{t-a}{b-a} \, . \end{array}$$

There is a natural equivalence relation on the set of oriented paths which will be very useful in the sequel. Its definition necessitates the introduction of the concept of **immediately retraced path**.

Def. 2.1.2 *A path* γ *is said to be immediately retraced if it can be written as* $\gamma = \prod_i \gamma_i {\gamma_i}^{-1}$*, for some paths* γ_i *in* M*.*

The following equivalence relation on the set of oriented paths in M was first introduced by Chen.

Def. 2.1.3 *Two oriented path* γ_1*,* γ_2 *are said to be* **elementary equivalent***,* $\gamma_1 \sim_{el} \gamma_2$*, if one is obtained from the other by composition with an immediately retraced path* γ*, i.e.* $\gamma_1 = \gamma_2 \gamma$*.*

It is worth noting that *in every elementary class of paths there is only one representative free from immediate retracing*, this representative is the path for which the immediately retraced path γ is the constant path in the ending point of the path itself (for all the other representatives γ is not trivial). This path is taken to be *the canonical representative* of the elementary equivalence class to which it belongs.

For the purposes of the loop quantization it will be seen that it is very important to have at one's hand the definition of finite graph embedded in a manifold, or simply graph, this needs the concepts of edge and vertex, which are introduced below.

Def. 2.1.4 *An* **edge** *in M is a continuous map $e : [0,1] \to M$ such that its restriction $\tilde{e} \equiv e|_{(0,1)}$ is an analytic embedding*[1] *of $(0,1)$ in M.*

The **vertexes** *of an edge are its starting and ending point, that is $e(0) \equiv s(e)$ and $e(1) \equiv t(e)$, also called* **source** *and* **target***, respectively.*

It is not possible to define the vertexes of an edge without choosing a parameterization because every representative in the class has, in general, different source and target, due to the reparameterization.

Def. 2.1.5 *A* **graph** *in M is the union of a finite family of images of edges intersecting only in their vertexes. A graph is said to be* **connected** *if the source of every edge is the target of another one.*

The usual symbol to denote a graph is Γ; the number of edges and vertexes of Γ will be indicated by E_Γ and V_Γ, respectively.

The following result is of essential importance:

Theorem 2.1.1 *For any piecewise analytic path (resp. loop) γ in M, its image γ^* is a graph (resp. connected graph) in M.*

Proof. Suppose first that γ is globally analytic, then its tangent map is injective but, at least, in a finite number of points, thus γ is a local embedding.

Moreover $[0,1]$ is compact and γ is continuous, thus γ^* is also compact and so it can be covered by a finite number of open analytic submanifolds of M given by some opportune arcs ${\gamma_i}^*$. The easiest way to obtain these arcs is to chose a partition of $[0,1]$ and to consider the restriction of γ to the subintervals.

[1] This means that $\tilde{e}$ is analytic and injective, with injective tangent map and $\tilde{e}^*$ is a submanifold of M w.r.t. the topology inherited by M.

Thanks to the assumption of analyticity, the arcs ${\gamma_i}^*$ intersect only in a finite number of point or they agree, hence the covering of γ^* can be refined by taking all the arcs which intersect themselves only in the extreme points.

This arcs clearly become the edges of the graph γ^* and their points of intersections become its vertexes.

If γ is only piecewise analytic, then the arguments above work again on every piece on which the path is analytic. □

In the proof of the theorem it has been shown that *the graph associated to a piecewise analytic path is not uniquely defined*, in fact it depends on the partition of $[0,1]$ which leads to the covering of γ^*. Obviously the graph is fixed when this partition is chosen.

The result just proved will be very useful in the sequel, hence we make the following: **the paths considered in the sequel will always assumed to be piecewise analytic**. The developing of a theory which uses piecewise smooth paths, instead of the analytic ones, is still under investigation.

Now we leave the generic paths and we put our attention on the loops, which will be indicated with α, β, ... All the loops will be based on the same point $\star \in M$, unless otherwise specified.

The first fact to stress is that one can easily define a law of composition between two loops α and β in this way

$$(\alpha\beta)(t) := \begin{cases} \alpha(2t) & \text{if t} \in [0, \frac{1}{2}] \\ \beta(2t-1) & \text{if t} \in [\frac{1}{2}, 1] \end{cases}$$

but this law doesn't give to the set of loops the structure of a group, in fact the composition between a loop α and its "inverse"

$$\begin{array}{cccl} \alpha^{-1}: & [0,1] & \longrightarrow & M \\ & t & \mapsto & \alpha^{-1}(t) := \alpha(1-t) \end{array}$$

is an immediately retraced loop, but it doesn't agree with the constant loop $\star$, which is obvious to take as the unit loop. Hence the set of all loops in M based on $\star$ with the composition law defined above is a semigroup with unit $\star$, usually denoted with $\Omega_\star(M)$, and not a group.

It is obvious that to give the group structure to the set of loops one has to take the quotient w.r.t. an equivalence relation which puts the loops of the type $\alpha\alpha^{-1}$ in the same class of the unit loop $\star$.

A natural equivalence relation to get this is the already defined elementary equivalence, in fact $\alpha\alpha^{-1} \sim_{el} \star \Leftrightarrow \exists\,\gamma$, immediately retraced path, such that $\alpha\alpha^{-1} = \star\gamma$, but such γ do exists and it is simply $\alpha\alpha^{-1}$!

Thanks to this arguments the following definition has perfectly sense.

Def. 2.1.6 *The set of all classes of elementary equivalence of loops in M based on $\star$ with composition law given by $[\alpha]_{el}[\beta]_{el} := [\alpha\beta]_{el}$ is a group called the* **group of loops** *and it is denoted by $L_\star(M)$.*

Obviously the unit of $L_\star(M)$ is $[\star]_{el}$ and the inverse of $[\alpha]_{el}$ is $[\alpha^{-1}]_{el}$. For shortness the class $[\alpha]_{el}$ will be identified with its canonical representative α.

The next step is to define the concepts of independent and simple loops.

Def. 2.1.7 *An oriented arc l of a path γ is said to be* **simple** *if $t_1 \neq t_2$ implies $l(t_1) \neq l(t_2)$, $\forall t_1, t_2 \in [0,1]$, i.e. l doesn't intersect itself (equivalently, the set $l^{-1}(x)$ is a singleton $\forall x \in l^*$).*

$\alpha \in L_\star(M)$ is said to be a **simple loop** *if in its elementary equivalence class there exists a representative which admits a simple arc.*

A finite family $\{\beta_i\}(i = 1, \ldots, n) \subset L_\star(M)$ is said to be **independent** *if every β_i admits a simple arc l_i such that $l_i \cap {\beta_j}^* = \emptyset$ $\forall i \neq j$, i.e. every loop of the family has a simple arc which doesn't intersect the images of the other loops of the same family. The β_i's are said to be* **independent loops**.

A very easy example of independent family of loops is the following: take $M \equiv \mathbb{R}^2$ and take γ_+,γ_- to be the upper and the lower half unit circle in $\mathbb{R}^2$, respectively. It is obvious that $\{\gamma_+, \gamma_-\}$ is an independent family. Instead the family $\{\gamma, \gamma_+, \gamma_-\}$, with $\gamma \equiv S^1$, is not independent, because every arc of γ intersect γ_+ or γ_- or both of them.

The importance of the notion of independence between loops is motivated by the next theorem.

Theorem 2.1.2 *Every $\alpha \in L_\star(M)$ is the composition of a finite family of simple independent loops.*

Proof. For simplicity fix the canonical representative α of an elementary class of loops in $L_\star(M)$ and a finite partition of $[0,1]$:

$$0 \leq a_1 < b_1 < \ldots < a_i < b_i < a_{i+1} < b_{i+1} \ldots < a_m < b_m \leq 1$$

such that $\alpha|_{[a_i,b_i]} \equiv l_i$ is an edge of the graph α^* and *every* edge of α^* is obtained in this way (we know that such a partition exists by the previous theorem).

For any fixed edge l_i, $1 < i < m$, take:

- a real analytic path ${q^i}_-$ connecting $\star$ with $\alpha(a_i)$, the source of l_i;
- a real analytic path ${q^i}_+$ connecting $\star$ with $\alpha(b_i)$, the target of l_i.

Take also $q^1{}_- \equiv \star$, $q^m{}_+ \equiv \star$ and $q^{i-1}{}_+ \equiv q^i{}_-$, $i = 2, \dots, m$.

Then the collection $\{\beta_{l_i} := q^i{}_- l_i (q^i{}_+)^{-1}\}(i = 1, \dots, m)$ is a finite family of simple independent loops, in fact every arc properly contained in each l_i doesn't intersect any other edge l_j, $j \neq i$, by definition of edge.

For every fixed i one has

$$\cdots \beta_{l_{i-1}} \beta_{l_i} \beta_{l_{i+1}} \cdots = \star \cdots l_{i-1} (q^i{}_-)^{-1} q^i{}_- l_i (q^{i+1}{}_-)^{-1} q^{i+1}{}_- l_{i+1} \cdots \star$$

hence the composition of powers of the loops β_{l_i} reconstructs α up to immediately retraced loops, i.e. it belongs to the same elementary equivalence class of α and so the theorem is proved. □

An easy, but useful, generalization of the last theorem is the following.

Corollary 2.1.1 *Given a finite family $\{\alpha_1, \dots, \alpha_r\} \subset L_\star(M)$, every loop of the family can be written as composition of loops belonging to a simple independent family $\{\beta_1, \dots, \beta_s\} \subset L_\star(M)$, for a certain integer $s \geq r$.*

Proof. To every loop α_j associate a graph Γ_j with edges $\{l_{j_i}\}$ with the property that the edges of two different graphs intersect themselves only in their vertexes or they coincide (we know that this is always possible by refining the parameterization of the loops). Now for every loop α_j define the simple independent loops β_{j_i} as in the proof of the theorem above, then the theorem follows from the same considerations. □

By iterating the arguments of the proofs above one has that if the family $\{\alpha_1, \dots, \alpha_r\} \subset L_\star(M)$ is extended to a family $\{\alpha_1, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_{r'}\} \subset L_\star(M)$ then there is another (in general different) family of simple independent loops $\{\beta'_1, \dots, \beta'_{s'}\} \subset L_\star(M)$ which decomposes every loop of the extended family and every loop of the independent family $\beta_1, \dots, \beta_s$.

Fixed a generic $\alpha \in L_\star(M)$, the more general decomposition of α as a product of powers of independent loops $\beta_1, \dots, \beta_m$ can be written as:

$$\alpha = \beta_1{}^{n_{1,1}} \cdots \beta_m{}^{n_{m,1}} \beta_1{}^{n_{1,2}} \cdots \beta_m{}^{n_{m,2}} \cdots \beta_1{}^{n_{1,k}} \cdots \beta_m{}^{n_{m,k}}$$

where $n_{i,j} \in \mathbb{Z}$, for every $i = 1, \dots, m$, $j = 1, \dots, k$. This decomposition will be used many times in the sequel.

If the elementary equivalence is the most natural, there are two other important equivalence relations between loops:

1. the *thin equivalence*;
2. the *holonomic equivalence.*

The first has a topological nature while the second has a geometrical character.

The quotient of the group of loops w.r.t. these relations gives rise to other groups and the structural relations between these groups are very interesting and useful.

The definition of thin equivalence relies on the concept of **thin loop**.

Def. 2.1.8 *A loop $\alpha \in L_\star(M)$ is said to be thin if it is homotopic to the constant loop $\star$ with a homotopy having image entirely contained in α^*.*

The definition is well posed because the immediately retraced loops are obviously thin, hence there is independence from the particular choice of the representative in the class $[\alpha]_{el}$.

Def. 2.1.9 *α,$\beta \in L_\star(M)$ are said to be* **thin equivalent***, $\alpha \sim_{th} \beta$, if there exists a thin loop γ such that $\alpha = \beta\gamma$.*

The set of all thin loops is easily recognized to be a normal subgroup of $L_\star(M)$ and so it is defined the quotient group $\mathcal{L}_\star(M) := L_\star(M)/Thin_\star(M) = \{\beta\gamma \mid \gamma \text{ thin}, \beta \in L_\star(M)\}$.

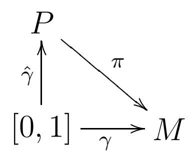
The definition of holonomic equivalence of loops requires the geometrical notion of holonomy, to which is entirely dedicated the following section.

2.2 Holonomy and holonomic equivalence of loops

In this section are presented the definitions and results which lead to the fundamental concept of holonomy. This one is then used to define the holonomic equivalence of loops.

A **horizontal lift** of a path $\gamma : [0,1] \to M$ is a path $\hat{\gamma} : [0,1] \to P$ satisfying the following conditions:

1. **lift condition**: the following diagram is commutative:



   i.e. $\pi(\hat{\gamma}_t) = \gamma_t$, for every $t \in [0,1]$;

2. **horizontal condition**: $\dot{\hat{\gamma}}_t \in H_{\gamma_t}(P)$, for every $t \in [0,1]$.

The notion of horizontal lift of a curve is closely related to that of a vector field, in fact if the vector field $\hat{X}$ on P is the horizontal lift of a vector field X on M, i.e. $\pi_*\hat{X} = X$, then the integral curve of $\hat{X}$ which starts in $p_0 \in P$ is the horizontal lift of the integral curve of X which starts in $\pi(p_0)$.

The most remarkable fact about horizontal lift of paths is expressed in the next theorem.

Theorem 2.2.1 *Let (P, M, G, π, R) be a principal fiber bundle and let γ be a smooth path in M. Then, fixed an arbitrary point $p_0 \in \pi^{-1}(\gamma_0)$, there exists one and only one horizontal lift $\hat{\gamma}$ of γ which starts in p_0.*

Proof. Let us start with the hypothesis that the bundle is trivial, then $P = M \times G$ and a lift of γ which starts in p_0 is suddenly individuated in the path $\eta_t := (\gamma_t, e)$. In fact, thanks to the global trivialization, one has $\pi(\eta_t) = pr_1(\gamma_t, e) = \gamma_t$, furthermore $\eta_0 = (\gamma_0, e) = p_0$.

If $\hat{\gamma}$ is another lift of γ which starts from p_0, then the lift condition implies that $\hat{\gamma}_t, \eta_t \in \pi^{-1}(\gamma_t)$, for every $t \in [0,1]$, hence, since the action of G on P is free and transitive on the fibers, $\hat{\gamma}$ must be of the form $\hat{\gamma}_t = \eta_t.g_t$, $\forall t \in [0,1]$, where $t \to g_t$ is a path in G starting from e. This initial condition is due to the fact that $\hat{\gamma}_0 = p_0 = \eta_0.g_0$.

Observe now that $\hat{\gamma}_t = \eta_t.g_t \equiv R(\eta_t, g_t)$, $\forall t \in [0,1]$, thus one can consider $\hat{\gamma}$ as the following composition of maps:

$$\begin{array}{ccccc} [0,1] & \longrightarrow & P \times G & \longrightarrow & P \\ t & \overset{(\eta\times g)}{\longmapsto} & (\eta_t, g_t) & \overset{R}{\longmapsto} & \eta_t.g_t \end{array}$$

so that $\hat{\gamma} = R \circ (\eta \times g)$.

The generalized Leibnitz rule enables to decompose the push-forward of this map as follows:

$$(R \circ (\eta \times g))_*(\dot{\eta}_t, \dot{g}_t) = (R_{g_t})_*(\dot{\eta}_t) + (R_{\eta_t})_*(\dot{g}_t).$$

Now, $\hat{\gamma}$ is a horizontal lift of γ if and only if $A(\hat{\gamma}_*(\dot{\eta}_t, \dot{g}_t)) = 0$, for every $t \in [0,1]$, i.e. if and only if

$$0 = A((R_{g_t})_*(\dot{\eta}_t)) + A((R_{\eta_t})_*(\dot{g}_t)),$$

but, thanks to the equivariance of A, the first term on the right hand side is precisely $Ad_{g_t^{-1}} A(\dot{\eta}_t)$.

Even the second term can be re-written in a more useful way, the functorial property of the push-forward implies $(R_{\eta_t})_* = (R_{\eta_t.g_t})_* \circ (L_{g_t^{-1}})_*$, and so

$$A((R_{\eta_t})_*(\dot{g}_t)) = A((R_{\eta_t.g_t})_*(L_{g_t^{-1}})_*(\dot{g}_t)) = (L_{g_t^{-1}})_*(\dot{g}_t)$$

where the last equality follows from the property of A to reproduce the generators of the fundamental vector fields.

The quantity $(L_{g_t^{-1}})_*(\dot{g}_t)$ is called **logarithmic derivative** and the mapping $t \mapsto (L_{g_t^{-1}})_*(\dot{g}_t)$ is a curve in $\mathfrak{g}$ because $(L_{g_t^{-1}})_*(\dot{g}_t) \in T_{g_t^{-1}g_t}G = T_eG \simeq \mathfrak{g}$.

The horizontal condition for the lift $\hat{\gamma}$ can now be re-written as

$$0 = Ad_{g_t^{-1}}A(\dot{\eta}_t) + (L_{g_t^{-1}})_*(\dot{g}_t)$$

or, more explicitly, using the fact that G is supposed to be a matrix group,

$$0 = {g_t}^{-1}A(\dot{\eta}_t)g_t + {g_t}^{-1}\dot{g}_t$$

i.e.

$$\dot{g}_t = -A(\dot{\eta}_t)g_t$$

which is a non-autonomous first-order ordinary linear differential equation in G, written in the normal form.

The non-autonomy of the equation depends on the fact that $A(\dot{\eta}_t)$ is an explicitly t-dependent (left-invariant) vector field on G.

Since the path $t \mapsto g_t$ must satisfy the initial condition $g_0 = e$, the horizontal condition for γ is actually equivalent to this Cauchy problem:

$$\begin{cases} \dot{g}_t = -A(\dot{\eta}_t)g_t \\ g_0 = e \end{cases}$$

hence the theorem of existence and uniqueness guarantees that this problem admits a unique solution.

If (P, M, G, π, R) is not trivial, then it is locally trivial, hence, fixed a gauge by the choice of a local section σ_α, an obvious lift of γ starting from p_0 is $\hat{\gamma} := \sigma_\alpha \circ \gamma$, in fact, by definition of section, $\pi(\hat{\gamma}_t) = \pi(\sigma_\alpha(\gamma_t)) = id_{U_\alpha}(\gamma_t) = \gamma_t$, for every $t \in [0, 1]$.

It follows that $\dot{\hat{\gamma}}_t = (\sigma_\alpha)_*(\dot{\gamma}_t)$ and so $\hat{\gamma}_t$ is horizontal if and only if $A(\dot{\hat{\gamma}}_t) = A((\sigma_\alpha)_*(\dot{\gamma}_t)) = {\sigma_\alpha}^*A(\dot{\gamma}_t) = A_\alpha(\dot{\gamma}_t) = 0$.

Hence the differential equation reached in the trivial case projects on U_α to give the differential equation:

$$\dot{g}_t = -A_\alpha(\dot{\gamma}_t)g_t.$$

When the image of the path γ is not contained in a single fibered chart U_α then it necessitates to consider a collection of fibered charts which cover the entire image of γ and to operate the same construction as above.

The solutions of the differential equations one reaches fit smoothly in the intersection of the charts, this is a consequence of the already cited transformation rule of the local connections for different choice of gauge, i.e.

$$A_\alpha = Ad_{g_{\alpha\beta}(x)^{-1}} A_\beta + g_{\alpha\beta}(x)^{-1}(g_{\alpha\beta})_* \qquad \forall x \in U_\alpha \cap U_\beta$$

from which it follows that $A_\alpha(\dot\gamma_t) = 0$, i.e $\dot g_t = -A_\alpha(\dot\gamma_t)g_t$, if and only if $0 = Ad_{g_{\beta\alpha}(x)^{-1}} A_\beta(\dot\gamma_t) + g_{\beta\alpha}(x)^{-1}(g_{\alpha\beta})_*$, $\forall x \in U_\alpha \cap U_\beta$, and so, in particular, this relation holds when $g_{\alpha\beta}(x) = g_t$, for every value of the parameter t such that $\gamma_t \in U_\alpha \cap U_\beta$, thus

$$0 = {g_t}^{-1} A_\beta(\dot\gamma_t) g_t + {g_t}^{-1}\dot g_t$$

i.e. $\dot g_t = -A_\beta(\dot\gamma_t)g_t$. In conclusion, the two differential equations are the same, or, equivalently, the left-invariant vector fields $A_\alpha(\dot\gamma_t)$ and $A_\beta(\dot\gamma_t)$ are the same for every t such that $\gamma_t \in U_\alpha \cap U_\beta$, which was the last thing to proof. □

2.2.1 Parallel transport in a principal fiber bundle

By varying the point p_0 in the fiber $\pi^{-1}(\gamma_0)$, one obtains a map from the fiber $\pi^{-1}(\gamma_0)$ to the fiber $\pi^{-1}(\gamma_1)$, defined obviously by

$$\begin{array}{rccc} \wp_{\gamma,A}: & \pi^{-1}(\gamma_0) & \longrightarrow & \pi^{-1}(\gamma_1) \\ & p & \mapsto & \wp_{\gamma,A}(p) := \hat\gamma_1 \end{array}$$

being $\hat\gamma$ the unique horizontal lift of γ which starts in p.

The map $\wp_{\gamma,A}$ is called **the parallel transport relative to the connection A along the path** γ and it depends both on A and γ.

The most important properties of the parallel transport are listed below.

Theorem 2.2.2 *Let A be a principal connection on a principal fiber bundle (P, M, G, π, R) and let γ be a smooth path in M, then the parallel transport induced by A along γ has the following properties:*

1. *$\wp_{\gamma,A}$ is unaffected by orientation preserving reparameterizations of γ;*
2. *$\wp_{\gamma,A}$ is equivariant, i.e. it commutes with R_g for every $g \in G$:*

$$\wp_{\gamma,A} \circ R_g = R_g \circ \wp_{\gamma,A}, \qquad \forall g \in G$$

more explicitly

$$\wp_{\gamma,A}(p.g) = \wp_{\gamma,A}(p).g \qquad \forall p \in P,\ \forall g \in G;$$

3. *$\wp_{\gamma,A}$ is a diffeomorphism of fibers and its inverse is given by the parallel transport induced by A along γ^{-1}:*

$$\wp_{\gamma,A}{}^{-1} = \wp_{\gamma^{-1},A};$$

4. *whenever the composite path $\gamma\eta$ is defined, $\wp_{\gamma\eta,A} = \wp_{\eta,A} \circ \wp_{\gamma,A}$ (the reason for the inversion of the order of γ and η is the opposite convention to compose paths and maps).*

Proof.
1. This follows from the fact that in the proof of the existence and uniqueness of the horizontal lift $\hat{\gamma}$ of γ starting from a fixed point only the direction of $\dot{\gamma}_t$, as tangent vector, has been used;

2. The equivariance of the parallel transport follows from this general feature of horizontal lifts:

Lemma 2.2.1 *If $\hat{\gamma}'$ and $\hat{\gamma}''$ are two arbitrary horizontal lifts of $\gamma : [0,1] \to M$, then it exists a fixed $g \in G$ such that*

$$\hat{\gamma}''_t = \hat{\gamma}'_t.g,$$

in particular, if γ' and γ'' starts in the same point, then $g \equiv e$.

Proof. In the proof of the theorem 2.2.1, η was an arbitrary lift of γ which started in the same point of the horizontal lift $\hat{\gamma}$, now, instead, $\hat{\gamma}'$ and $\hat{\gamma}''$ are both horizontal lifts of γ, but they doesn't necessary start in the same point.

By the way, the lift condition imposes again that there must be a path $t \mapsto g_t$ in G such that $\hat{\gamma}''_t = \hat{\gamma}'_t.g_t$, $\forall t \in [0,1]$, hence to prove the thesis of the lemma it suffices to prove that $t \mapsto g_t$ is a constant map.

But this is very easy, in fact by using the generalized Leibnitz rule to the identity $\hat{\gamma}''_t = \hat{\gamma}'_t.g_t$ and applying the connection A to both members one gets:

$$A(\dot{\hat{\gamma}}''_t) = Ad_{g_t^{-1}} A(\dot{\hat{\gamma}}'_t) + (L_{g_t^{-1}})_*(\dot{g}_t)$$

but $\hat{\gamma}'$ and $\hat{\gamma}''$ are both horizontal, thus $A(\dot{\hat{\gamma}}''_t) = A(\dot{\hat{\gamma}}'_t) = 0$, for every $t \in [0,1]$, hence

$$(L_{g_t^{-1}})_*(\dot{g}_t) = 0.$$

Now, $(L_{g_t^{-1}})_*$ is a linear isomorphism and so it vanishes only on the null tangent vector, i.e. $\dot{g}_t = 0$ $\forall t \in [0,1]$ and so $t \mapsto g_t$ is a constant path in G. □

Thanks to this result the proof of the equivariance of the parallel transport is very easy, in fact it suffices to specialize $\hat{\gamma}'$ to be the horizontal lift of γ which starts in $p_0 \in \pi^{-1}(\gamma_0)$ and $\hat{\gamma}''$ to be the horizontal lift of γ which starts in $p_0.g \in \pi^{-1}(\gamma_0)$, then

$$p_0.g = \hat{\gamma}_0'' = \hat{\gamma}_0'.g = p_0.g$$

and, thanks to the lemma, $\hat{\gamma}_1'' = \hat{\gamma}_1'.g$ (the *same* g).

By definition $\wp_{\gamma,A}(p_0.g) = \hat{\gamma}_1'' = \hat{\gamma}_1'.g = \wp_{\gamma,A}.g$.

3. First of all it has to be proved that $\wp_{\gamma,A}$ is a bijection with inverse given by $\wp_{\gamma^{-1},A}$. This is trivial, in fact $\hat{\gamma}_0^{-1} = \hat{\gamma}_1$ and $\hat{\gamma}_1^{-1} = \hat{\gamma}_0$ thus

$$\wp_{\gamma^{-1},A}(\wp_{\gamma,A}(p_0)) = \wp_{\gamma^{-1},A}(\hat{\gamma}_1) = \wp_{\gamma^{-1},A}(\hat{\gamma}_0^{-1}) = \hat{\gamma}_1^{-1} = \hat{\gamma}_0 = p_0$$

where in the third passage it has been used the definition of the parallel transport along γ^{-1}.

Analogously one proves that $\wp_{\gamma,A}(\wp_{\gamma^{-1},A}(p_0)) = p_0$.

This shows that $\wp_{\gamma,A}$ is a bijection between fibers, furthermore it is constructed by smooth horizontal lifts of smooth curve and so it is itself smooth with smooth inverse, i.e. a diffeomorphism.

4. As a consequence of our definition of composition of paths, $\widehat{\gamma\eta}_0 = \hat{\gamma}_0 \equiv p_0 \in \pi^{-1}(\gamma_0)$, $\widehat{\gamma\eta}_1 = \hat{\eta}_1 \in \pi^{-1}(\eta_1)$, $\gamma_1 = \eta_0$ and $\hat{\gamma}_1 = \hat{\eta}_0$, hence, by definition, $\wp_{\gamma\eta,A}(p_0) = \hat{\eta}_1$.

Furthermore, $\wp_{\eta,A}(\wp_{\gamma,A}(p_0)) = \wp_{\eta,A}(\hat{\gamma}_1) = \wp_{\eta,A}(\hat{\eta}_0) := \hat{\eta}_1$.

Thus $\wp_{\gamma\eta,A}(p_0) = \hat{\eta}_1 = \wp_{\eta,A}(\wp_{\gamma,A}(p_0))$, for every fixed $p_0 \in \pi^{-1}(\gamma_0)$. □

2.2.2 Holonomy and holonomy groups

If, instead of generic paths, one fixes the attention on loops, always denoted as α or β in the sequel, then the concept of holonomy arises naturally.

In fact, for a loop $\alpha \in \Omega_\star(M)$, the starting and the ending point are both $\star$, thus the parallel transport $\wp_{\alpha,A}$ is an automorphism of the fiber $\pi^{-1}(\star)$.

By varying the loop α in $\Omega_\star(M)$ one gets the set

$$\mathcal{H}_\star := \{\wp_{\alpha,A} \mid \alpha \in \Omega_\star(M)\}$$

which, thanks to the properties of the parallel transport examined above, becomes a group when it is endowed with the composition law

$$\wp_{\alpha,A} \circ \wp_{\beta,A} = \wp_{\beta\alpha,A}$$

so that $\wp_{\alpha,A}{}^{-1} = \wp_{\alpha^{-1},A}$ and the unity is $\wp_{\star,A}$.

The group $\mathcal{H}_\star$ is called **the holonomy group of the connection A in the point $\star$.**

If $\Omega^0_\star(M)$ denotes the set of the loops homotopic to the constant loop $\star$ in M, then the subgroup of $\mathcal{H}_\star$ defined by

$$\mathcal{H}^0_\star := \{\wp_{\alpha,A} \mid \alpha \in \Omega^0_\star(M)\}$$

is called the **restricted holonomy group of the connection A in the point $\star$.**

Obviously, if M is simply connected then $\mathcal{H}_\star \equiv \mathcal{H}^0_\star$.

The following discussion will be focused only on the holonomy group, analogous results holds even for the restricted holonomy groups.

The first remarkable fact about the holonomy group is that it can be conveniently represented as a subgroup of the structural group G.

Theorem 2.2.3 *Let $\mathcal{H}_\star$ be the holonomy group of a connection A on a principal fiber bundle $P(M,G)$ in the point $\star \in M$. Then, fixed a point $p \in \pi^{-1}(\star)$, the map*

$$\begin{array}{rccl} j_p: & \mathcal{H}_\star & \longrightarrow & \mathcal{H}_p \subset G \\ & \wp_{\alpha,A} & \mapsto & j_p(\wp_{\alpha,A}) := H_A(\alpha) \end{array}$$

with $H_A(\alpha)$ defined by the equation

$$\wp_{\alpha,A}(p) := p.H_A(\alpha)^{-1}$$

is a group isomorphism.

Proof. Since $\wp_{\alpha,A}$ is a diffeomorphism of $\pi^{-1}(\star)$ into itself and since $\pi^{-1}(\star)$ is a homogeneous space for G, then it exists one and only one $g \in G$ such that $\wp_{\alpha,A}(p) = p.g$, in the thesis of the theorem this g is indicated by $H_A(\alpha)^{-1}$.

This observation proofs the bijective nature of j_p and so to prove the theorem it remains only to show that j_p preserves the group structures of $\mathcal{H}_\star$ and G, i.e. that

$$j_p(\wp_{\alpha\beta,A}) = H_A(\alpha)H_A(\beta).$$

This is very easy, in fact

$$\begin{array}{rll} \wp_{\alpha\beta,A}(p) & = (\wp_{\beta,A} \circ \wp_{\alpha,A})(p) & = \wp_{\beta,A}(\wp_{\alpha,A}(p)) = \\ & = \wp_{\beta,A}(p.H_A(\alpha)^{-1}) & = \wp_{\beta,A}(R_{H_A(\alpha)^{-1}}(p)) \\ & = (\wp_{\beta,A} \circ R_{H_A(\alpha)^{-1}})(p) & = \text{equivariance of } \wp_{\beta,A} = \\ & = (R_{H_A(\alpha)^{-1}} \circ \wp_{\beta,A})(p) & = R_{H_A(\alpha)^{-1}}(\wp_{\beta,A}(p)) = \\ & = R_{H_A(\alpha)^{-1}}(p.H_A(\beta)^{-1}) & = (R_{H_A(\alpha)^{-1}} \circ R_{H_A(\beta)^{-1}})(p) = \\ & = R_{H_A(\alpha)^{-1}H_A(\beta)^{-1}}(p) & = R_{(H_A(\beta)H_A(\alpha))^{-1}}(p) = \\ & = p.(H_A(\alpha)H_A(\beta))^{-1} & \end{array}$$

$\square$

From the proof of this last theorem it is obvious that the inversion of $H_A(\alpha)$ in the relation $\wp_{\alpha,A}(p) := p.H_A(\alpha)^{-1}$ is essential in order to make j_p into a isomorphism of groups, if one doesn't invert $H_A(\alpha)$ in the above relation then j_p becomes an anti-isomorphism of groups.

The image of j_p in G is called the **holonomy group of the connection A in the point $p \in P$** and it is denoted by $\mathcal{H}_p$, in particular the element $H_A(\alpha) \in G$ is called the **holonomy of A with respect to the loop α**.

Alternatively on can define $\mathcal{H}_\star$ to be the image of the map which sends every loop α in the holonomy of A with respect to it, i.e.

$$\begin{array}{rccc} H_A: & \Omega_\star(M) & \longrightarrow & G \\ & \alpha & \mapsto & H_A(\alpha). \end{array}$$

Even though $H_A(\alpha\beta) = H_A(\alpha)H_A(\beta)$, for every couple of loops α and β, H_A is not a group homomorphism because $\Omega_\star(M)$ is not a group! It becomes a group homomorphism when $\Omega_\star(M)$ is quotiented with respect to an equivalence relation which doesn't affect the parallel transport. This topic will be discussed later in this chapter.

Note also that the map

$$\begin{array}{rccc} \wp_A: & \Omega_\star(M) & \longrightarrow & \mathcal{H}_\star \\ & \alpha & \mapsto & \wp_{\alpha,A} \end{array}$$

is the only one which makes the following diagram

$$\begin{array}{ccc} \Omega_\star(M) & & \\ {\scriptstyle \wp_A}\Big| & \searrow^{H_A} & \\ \mathcal{H}_\star & \xrightarrow[j_p]{} & \mathcal{H}_p \end{array}$$

commutative.

2.2.3 Computation of holonomies

Remember the holonomies born as solutions in the ending value of the parameter $t = 1$ of the differential equation

$$\dot{g}_t = A(\dot{\alpha}_t)g_t$$

i.e. $H_A(\alpha) = g_1$ (and not g^{-1} because we have changed the sign to the equation!). Here A is identified with one of its local expressions to have a clearer notation.

It has already been remarked that the differential equation above is a non-autonomous first-order linear differential equation, luckily there is a well established theory to solve this kind of equations, in the sequel are presented the most important features of this theory.

Let E be a Banach space with norm $\| \ \|$ and let $A \in \mathcal{B}(E)$, where $\mathcal{B}(E)$ denotes the algebra of the bounded linear operators from E into itself. By virtue of the inequality $\|A^n\| \leq \|A\|^n$, for every $n \in \mathbb{N}$, the series

$$\sum_{n=0}^{+\infty} \frac{A^n}{n!}$$

is absolutely convergent in $\mathcal{B}(E)$ and thus defines a bounded linear operator on E which is called the exponential of A:

$$e^A := \sum_{n=0}^{+\infty} \frac{A^n}{n!}$$

because it agrees with the usual exponential when $E \equiv \mathbb{R}$ or $\mathbb{C}$.

A well known property of the exponential of A is that, as the ordinary real exponential, it satisfies:

$$\frac{d}{dt}(e^{tA}) = Ae^{tA} \qquad \forall t \in [0,1]$$

hence the solution of the Cauchy problem for the curve $\xi : [0,1] \to E$ defined by

$$\begin{cases} \frac{d}{dt}(\xi_t) = A(\xi_t) & t \in [0,1] \\ \xi_0 = x \end{cases}$$

is given by $\xi_t = e^{tA}x$, $\forall t \in [0,1]$.

Since the operator A is fixed in $\mathcal{B}(E)$, the differential equation appearing in the Cauchy problem above is **autonomous**.

If one consider instead of a fixed $A \in \mathcal{B}(E)$ an operator-valued curve, denoted again with A for simplicity, $A : [0,1] \to \mathcal{B}(E)$, then the equation

$$\frac{d}{dt}(\xi_t) = A_t \xi_t \quad t \in [0,1]$$

becomes non autonomous and its solution is no more given by the exponential of A but by the so-called **temporary-ordered exponential** or **path-ordered exponential** of the operator-valued curve A.

To easily define this object it's convenient to put the attention of the case in which $t \mapsto A_t$ is a step function, i.e. there exists a finite partition

$0 = t_0 < \ldots < t_n = 1$ of $[0,1]$ such that $A|_{(t_{i-1},t_i)}$ is constant $\forall i = 1, \ldots, n$, i.e., $\forall t \in (t_{i-1}, t_i)$ $A_t \equiv A^i$, fixed operator of $\mathcal{B}(E)$, for every $i = 1, \ldots, n$.

By defining the norm of the step function A as

$$\|A\| := \sup_{i=1,\ldots,n} \|A^i\|$$

it can be proved that the completion of the space of these step function with respect to the topology induced by the norm above, the so-called **space of the regulated functions**, contains in particular all the continuous (and hence all the smooth) curves from $[0,1]$ to $\mathcal{B}(E)$.

Thus if one defines the temporary-ordered exponential for the step functions, then the definition can easily be extended to all the continuous or smooth maps from $[0,1]$ to $\mathcal{B}(E)$ via uniform limit.

Now, given a step function $A : [0,1] \to \mathcal{B}(E)$, its temporary-ordered exponential is the operator of $\mathcal{B}(E)$ defined as follows:

$$T\exp \int_0^1 A_t dt := e^{\triangle t_n A^n} \cdots e^{\triangle t_1 A^1}$$

where $\triangle t_i := t_i - t_{i-1}$, $i = 1, \ldots, n$.

Observe that every factor which appears on the right hand side is the usual exponential of operators, since A^i is a fixed operator in the interval (t_{i-1}, t_i).

The reason for the name 'temporary-ordered exponential' is obviously due to the fact that the first operator which acts (i.e. the one at the right) is that relative to the smallest value of the parameter t, the subsequent operators follows by increasing values of t, **chronologically**.

The inequality

$$\|T\exp \int_0^1 A_t dt - T\exp \int_0^1 B_t dt\| \leq e^{\max\{\|A\|,\|B\|\}} \|A - B\|$$

holds whenever A and B are step functions from $[0,1]$ to $\mathcal{B}(E)$, hence the mapping $A \to T\exp \int_0^1 A_t dt$ is uniformly continuous on every bounded subset of step functions and so it is uniquely extendible to the space of the regulated functions from $[0,1]$ to $\mathcal{B}(E)$ via uniform limit: if $\{A_n\}(n \in \mathbb{N})$ is a sequence of step functions uniformly convergent to the continuous (or smooth) operator-valued curve A, then

$$T\exp \int_0^1 A_t dt := \lim_{n\to+\infty} T\exp \int_0^1 A_n(t) dt.$$

The most important properties of the $T\exp$ are listed below:

1. $\left(T\exp\int_0^1 A_t dt\right)^{-1} = T\exp\int_1^0 A_t dt$;

2. $\left(T\exp\int_s^1 A_t dt\right)\left(T\exp\int_0^s A_t dt\right) = T\exp\int_0^1 A_t dt$, observe the order, consistent with the chronological growth of t;

3. if $t \mapsto A_t$ is a continuous curve in $\mathcal{B}(E)$, then the Cauchy problem for the path $\xi : [0,1] \to E$ given by
$$\begin{cases} \dot{\xi}_t = A_t(\xi_t) \\ \xi_0 = x \end{cases}$$
is solved in a unique way by $\xi_t = \left(T\exp\int_0^t A_s ds\right) x$, for every $t \in [0,1]$.

Finally let us specialize this dissertation to the computation of holonomies: the Cauchy problem for the curve $t \mapsto g_t$ given by
$$\begin{cases} \dot{g}_t = (A(\dot{\alpha}_t))g_t \\ g_0 = I \end{cases}$$
(where now $t \mapsto A(\dot{\alpha}_t)$ is a curve in $\mathfrak{g}$), is solved in a unique way by
$$g_t = \left(T\exp\int_0^t A(\dot{\alpha_s})ds\right) I$$
and so the holonomy of A along the loop α can be written as
$$g_1 = H_A(\alpha) = T\exp\int_0^1 A(\dot{\alpha_s})ds$$
where we have omitted I, the identity matrix. By expanding the operator exponentials appearing in the expression of $T\exp$ in power series, one reaches a formula which is very useful for approximated calculus of the holonomies, often used in gauge theory on lattice:
$$\begin{aligned} H_A(\alpha) = \; & I + \int_0^1 A_{t_1} dt_1 + \int_0^1 dt_1 \int_0^{t_1} A_{t_2} A_{t_1} dt_2 + \ldots + \\ & + \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} A_{t_n} \cdots A_{t_1} dt_n + \ldots \, , \end{aligned}$$
or, equivalently:
$$H_A(\alpha) = \sum_{n=0}^{+\infty} \int_{0 \le t_1 \le \ldots \le t_n \le 1} A_{t_n} \cdots A_{t_1} dt_1 \cdots dt_n .$$

In the relativistic theories can be misleading to interpret t as the time-parameter, hence the time-ordered exponential in such theories is more precisely called **path-ordered exponential** and it is indicated with $\mathcal{P}\exp$, so that the holonomy $H_A(\alpha)$ is denoted with
$$H_A(\alpha) = \mathcal{P}\exp\oint_\alpha A.$$

2.2.4 The holonomy map

From the properties 1. and 3. of the theorem 2.2.2 it follows that the holonomy of every loop belonging to the same elementary equivalence class is the same, so that is well defined the map which assigns to $\alpha \in L_\star(M)$ its holonomy w.r.t. a fixed connection A, i.e.:

$$\begin{array}{rccc} H_A: & L_\star(M) & \longrightarrow & G \\ & \alpha & \mapsto & H_A(\alpha) \end{array}$$

this is called **holonomy map**.

From the property 3. and 4. of the theorem 2.2.2 one gets the following properties of the holonomy map:

1. $H_A(\alpha\beta) = H_A(\alpha)H_A(\beta)$ "**factorization property**" ;
2. $H_A(\alpha^{-1}) = H_A(\alpha)^{-1}$.

Furthermore if A is a flat connection then $H_A(\alpha) = e$, for every $\alpha \in L_\star(M)$.

From the factorization property it follows that **the holonomy map is a homomorphism from the group of loops to the gauge group**.

Thus, for **unitary gauge theories**, where the gauge group G is a subgroup of the unitary group $U(N)$, H_A **realizes an unitary representation of** $L_\star(M)$.

Now the third equivalence relation on the set of loops can be introduced.

Def. 2.2.1 *Two loops α and β are said to be* **holonomy equivalent** *if they have the same holonomy w.r.t. every connection, i.e.*

$$H_A(\alpha) = H_A(\beta) \qquad \forall A \in \mathcal{A}\,.$$

The holonomic equivalence will be indicated by $\alpha \sim_{hol} \beta$ and a class of holonomic equivalence of loops will be called **hoop**.

The quotient of $L_\star(M)$ w.r.t. the holonomic equivalence gives rise to a group called the **hoop group** *and indicated with*

$$\mathcal{H}_\star(M,G) \equiv L_\star(M)/\sim_{hol}\ .$$

The term "hoop" is the abbreviation of "holonomic equivalence class of loops". By definition, to every representative of a given hoop there corresponds the same holonomy, hence the holonomy map factorizes to a homomorphism between $\mathcal{H}_\star(M,G)$ and G by the position: $H_A|_{\mathcal{H}_\star(M,G)}([\alpha]_{hol}) := H_A(\alpha)$, for an arbitrary representative α in the hoop $[\alpha]_{hol}$.

In the following, where there won't be risk of confusion, the notation $[\alpha]_{hol}$ will be abbreviated by α.

2.3 Relations between the three loop groups: $L_\star(M)$, $\mathcal{L}_\star(M)$ and $\mathcal{H}_\star(M,G)$

The analysis of the relations between the three groups $L_\star(M)$, $\mathcal{L}_\star(M)$ and $\mathcal{H}_\star(M,G)$ begins with the simple observation that *the elementary and thin equivalence have a topological nature, hence they are independent from the choice of the gauge group*, instead *the holonomic equivalence has geometrical character, thus it can depend on the choice of the gauge group, and in fact, as will be proved later, it does.* This is the reason why in the symbol chosen to denote the hoop group it appears G.

A second trivial observation is that the elementary equivalence implies both the thin and the holonomic equivalence, the problem is to understand if, or under what conditions, the converse is true.

A key step in the analysis of this problem is the introduction of the **holonomic kernel** of the principal bundle P: this is defined as the intersection of the kernels of all the holonomy maps, i.e.

$$K(P(M,G)) := \bigcap_{A\in\mathcal{A}} Ker(H_A)\,.$$

Thanks to the fact that the holonomy maps are homomorphisms, the holonomic kernel $K(P(M,G))$ is obviously a normal subgroup of $L_\star(M)$ so that one can consider the quotient group $L_\star(M)/K(P(M,G))$.

Theorem 2.3.1 $\mathcal{H}_\star(M,G) \equiv L_\star(M)/K(P(M,G))$

Proof.

$\subseteq$: if α and β are two elements of $L_\star(M)$ such that $H_A(\alpha) = H_A(\beta)$ for every $A \in \mathcal{A}$, then $H_A(\beta\alpha^{-1}) = e\ \forall A \in \mathcal{A}$, but this implies that $\beta\alpha^{-1} \equiv k \in K(P(M,G))$, hence $\beta = k\alpha$, i.e. α and β belong to the same equivalence class in the quotient group $L_\star(M)/K(P(M,G))$.

$\supseteq$: conversely, two generic representatives of a coset in $L_\star(M)/K(P(M,G))$ have the form $k\alpha$ and $k'\alpha$, with $k,k' \in K(P(M,G))$ and $\alpha \in L_\star(M)$, then $H_A(k\alpha) = H_A(k)H_A(\alpha) = eH_A(\alpha) = H_A(k')H_A(\alpha) = H_A(k'\alpha)$ and this holds for every $A \in \mathcal{A}$ because k and k' belong to $K(P(M,G))$. □

Note also that, if the fixed point $p_0 \in P_\star$ is changed, $K(P(M,G))$ doesn't change because $Ker(H_A)$ is insensible to changes of p_0.

The relation between $K(P(M,G))$ and $Thin_\star(M)$ is stated in the next theorem.

Theorem 2.3.2 *The holonomic kernel of $P(M,G)$ contains the thin loops: $Thin_\star(M) \subseteq K(P(M,G))$. Hence, every thin loop α has trivial holonomy w.r.t. any connection: $H_A(\alpha) = e \ \forall A \in \mathcal{A}$.*

Proof. Suppose that the thin loop α is an embedding of $[0,1]$ in M and consider the homotopy map $h : [0,1] \times [0,1] \to M$ which makes it thin, then, by definition:

$$\begin{cases} h(t,0) = \alpha(t) & \forall t \in [0,1]; \\ h(t,1) = \star & \forall t \in [0,1]; \\ h(t,s) \in \alpha^* & \forall (t,s) \in [0,1] \times [0,1]. \end{cases}$$

If h^*P is the pull-back of the principal fiber bundle P w.r.t. the homotopy map h, then the canonical morphism $h^* : h^*P \to P$ transforms horizontal lifts in h^*P w.r.t. the connection h^*A along the path $\gamma : [0,1] \to \partial([0,1] \times [0,1])$ (the counterclockwise oriented unit square in the t-s plane) into horizontal lifts in P w.r.t. the connections A along the path $h \circ \gamma$.

Observe that the image of h is a loop in M, thus it is a monodimensional submanifold of M, hence the connection h^*A is flat and so

$$e = H_{h^*A}(\gamma) = H_A(h \circ \gamma).$$

Since γ takes values on the boundary of $[0,1] \times [0,1]$ and since α is supposed to be thin, $h \circ \gamma$ is precisely $\alpha\star = \alpha$ (remember in fact that, by definition of thin loop, the image of h remains in α^*, this step furnishes the explanation for that request). Thus $H_A(h \circ \gamma) = (\alpha) = e$, and this holds for every arbitrary connection A, hence α belongs to the holonomic kernel.

Thanks to the fact that α is assumed to be piecewise analytic, if it isn't an embedding, then it is the composition of embeddings and so the arguments above work again to prove the theorem. □

As an immediate consequence of this theorem, one gets the inclusions chain between the loop groups:

$$\mathcal{H}_\star(M,G) \quad \mathcal{L}_\star(M) \quad L_\star(M) \text{ for every } G.$$

At first sight the holonomic kernel seems to depend on the entire structure of the principal bundle $P(M,G)$, actually it depends only on M and G and this fact will be very useful to obtain the relations between the loop groups. The independence of the holonomic kernel on the principal bundle is an easy consequence of the following result, which is one of the most important of all this chapter.

Theorem 2.3.3 *For every finite family* $\{g_1, \dots, g_n\} \subset G$*, there is a finite independent family of loops* $\{\beta_1, \dots, \beta_n\} \subset L_\star(M)$ *and a connection* A *such that*

$$g_i = H_A(\beta_i) \qquad i = 1, \dots, n.$$

This fact will be called **interpolation property**.

Proof. Fix a reference connection $A_0 \in \mathcal{A}$, then, thanks to the affine nature of $\mathcal{A}$, every connection is of the form $A_0 + \Omega$, where Ω is a 1-form on M with values in the associated bundle $P[\mathfrak{g}]$ (see [32]). Choose $\Omega \equiv X\omega$, with $X \in \mathfrak{g}$ and ω a scalar 1-form on M with support in some trivializing neighborhood. Applying formula 5.2 in [14] one obtains:

$$H_A(\alpha) = \mathcal{P}\exp \oint_\alpha (A_0 + X\omega) = \left(\mathcal{P}\exp \oint_\alpha X\omega\right) H_{A_0}(\alpha) = (\exp X\omega(\alpha)) H_{A_0}(\alpha)\,.$$

Define also

$$g_i^0 := H_{A_0}(\beta_i)\,,\ i = 1, \dots, n\,.$$

Thanks to the hypothesis that G is a compact connected Lie group, the exponential map $\exp : \mathfrak{g} \to G$ is surjective and so every element of G is the product of some element in the range of exp.

Thus there exist $\{X_{i,j_i}\}(j_i = 1, \dots, k_i) \subset \mathfrak{g}$ such that the element $g_i(g_i^0)^{-1}$ factorizes as

$$g_i(g_i^0)^{-1} = \exp X_{i,1}\, \exp X_{i,2} \cdots \exp X_{i,k_i}\,.$$

Thanks to the hypothesis of independence, it is possible to select a family of mutually disjoint simple arcs l_{i,j_i} contained in l_i, for $1 \le j_i \le k_i$, and a family of scalar 1-forms ω_{i,j_i} with trivializing supports U_i such that

- $l_{i,j_i} = \beta_i \cap U_i$;
- $\omega_{i,j_i}(l_{i,j_i}) = 1$;
- $U_i \cap \beta_\imath = \emptyset$ whenever $i \neq \imath$.

Finally take

$$A \equiv A_0 + \sum_{i,j_i} X_{i,j_i}\omega_{i,j_i}$$

then, by induction

$$H_A(\beta_i) = \exp X_{i,1} \exp X_{i,2} \cdots \exp X_{i,k_i} H_{A_0}(\beta_i) = g_i(g_i^0)^{-1} g_i^0 = g_i$$

for every $i = 1, \dots, n$ and so the theorem is proved. □

Corollary 2.3.1 *For every homomorphism $H : L_\star(M) \to G$ and every finite family of loops $\{\alpha_1, \dots, \alpha_n\} \subset L_\star(M)$ there exists a connection A such that*

$$H(\alpha_i) = H_A(\alpha_i) \qquad i = 1, \dots, n\,.$$

In other words, the action of an algebraic homomorphism from $L_\star(M)$ to G on a finite family of loops is identical to that of a certain holonomy map.

Proof. It is sufficient to factorize every loop of the finite family $\{\alpha_1, \dots, \alpha_n\}$ as the product of certain independent loops $\{\beta_j\}$. In fact, by defining $g_j := H(\beta_j)$ and applying the last theorem there exists a connection $A \in \mathcal{A}$ such that $H_A(\beta_j) = g_j$ for every j. As a consequence of the factorization property of the holonomy map it follows that $H(\alpha_i) = H_A(\alpha_i)$ $i = 1, \dots, n$. □

Corollary 2.3.2 *The holonomic kernel of the principal bundle $P(M, G)$ and the hoop group $\mathcal{H}_\star(M, G)$ don't depend on the structure of the principal fiber bundle, but only on M and G.*

Proof. The intersection of the kernels of the holonomy maps agrees with that of the algebraic homomorphisms $H : L_\star(M) \to G$, but this depends only on M and G and not on the principal bundle P. The analogue assertion on $\mathcal{H}_\star(M, G)$ follows immediately from the fact that $\mathcal{H}_\star(M, G)$ is the quotient of the loop group on the holonomic kernel. □

As a consequence the holonomic kernel will be denoted with $K(M, G)$ in the sequel.

Corollary 2.3.3 *There is a bijection between the homomorphisms from the loop group to G and those from the hoop group to G, i.e.*

$$Hom(L_\star(M), G) \simeq Hom(\mathcal{H}_\star(M, G), G)\,.$$

Proof.

- $Hom(\mathcal{H}_\star(M, G), G)$ $Hom(L_\star(M), G)$: obviously every homomorphism $\tilde{H} \in Hom(\mathcal{H}_\star(M, G), G)$ can be obtained from a homomorphism $H \in Hom(L_\star(M), G)$ by restriction;

- $Hom(L_\star(M), G)$ $Hom(\mathcal{H}_\star(M, G), G)$: let $\pi : L_\star(M) \to \mathcal{H}_\star(M, G)$, $\alpha \mapsto [\alpha]_{hol}$, denote the canonical projection of $L_\star(M)$ onto $\mathcal{H}_\star(M, G)$, then, for every $H \in Hom(L_\star(M), G)$ and $\tilde{H} \in Hom(\mathcal{H}_\star(M,G), G)$, the following diagram is commutative in the category of groups

$$\begin{array}{ccc} L_\star(M) & = & L_\star(M) \\ \pi \downarrow & & \downarrow H \\ \mathcal{H}_\star(M,G) & \xrightarrow[\tilde{H}]{} & G \end{array}$$

hence every H is surjectively induced by $\tilde{H} \in Hom(\mathcal{H}_\star(M,G), G)$ through the relation $H = \tilde{H} \circ \pi$.

□

Before proceeding it is worth remembering two concepts from group theory and a standard result about Lie groups: for every element g of an arbitrary group G, the **order** of g is the smallest integer n such that $g^n = e_{\mathsf{G}}$, if this integer doesn't exist then g is said to have **infinite order**. Moreover, if every element of G has infinite order then it is said a **torsion free group**. By converse, if every element of G has finite order then it is said a **torsion group**. A Lie group is never a torsion group, in fact it can be proved that *in every Lie group there exists, at least, one element of infinite order.*

The **commutator subgroup** of G, $Comm(G)$, is the normal subgroup of G generated by all the elements of G of the form $ghg^{-1}h^{-1}$. Every $a \in Comm(\mathsf{G})$ has the expression:

$$a = ({a_1}^{n_{1,1}}{a_2}^{n_{2,1}} \cdots {a_m}^{n_{m,1}})({a_1}^{n_{1,2}}{a_2}^{n_{2,2}} \cdots {a_m}^{n_{m,2}}) \cdots ({a_1}^{n_{1,k}}{a_2}^{n_{2,k}} \cdots {a_m}^{n_{m,k}})$$

with $a_1, \ldots, a_m \in \mathsf{G}$ and with $n_{i,j} \in \mathbb{Z}$, $i = 1, \ldots, m$, $j = 1, \ldots, k$ satisfying:

$$\sum_{j=1}^{k} n_{i,j} = 0$$

for every $i = 1, \ldots, m$.

Theorem 2.3.4 *The following relations hold:*

1. $K(M,G) \subseteq Comm(L_\star(M))$*;*
2. *if G is Abelian then* $K(M,G) = Comm(L_\star(M))$.

Proof. 1. Suppose that α belongs to the holonomic kernel and write it as the composition of powers of a certain simple independent family $\{\beta_i\}$:

$$\alpha = {\beta_1}^{n_{1,1}} \cdots {\beta_m}^{n_{m,1}} \cdots {\beta_1}^{n_{1,k}} \cdots {\beta_m}^{n_{m,k}} .$$

Thanks to what remembered before, to prove that α belongs to the commutator subgroup of $L_\star(M)$ it is sufficient to prove that

$$\sum_{j=1}^{k} n_{i,j} = 0 \qquad \forall i = 1, \dots, m\,.$$

By absurd, if there exists an index $i_0 \in \{1, \dots, m\}$ such that

$$N_{i_0} := \sum_{j=1}^{k} n_{i_0,j} \neq 0$$

then, thanks to the fact that a Lie group is never a torsion group, there exists $g \in G$ such that $g^{N_{i_0}} \neq e$.

Furthermore, thanks to the interpolation property, assigned m elements $\{g, e, \dots, e\}$ in G, there exists a connection A such that $H_A(\beta_{i_0}) = g$ and $H_A(\beta_i) = e$, for every other $i \neq i_0$.

But this is absurd, because, by virtue of the factorization property one has that $H_A(\alpha) = g^{N_{i_0}} \neq e$, against the assumption that α belongs to the holonomic kernel.

2. If now G is an Abelian group, then for every $\alpha, \beta \in L_\star(M)$ and for all $H \in Hom(L_\star(M), G)$ one has:

$$\begin{aligned} H(\alpha\beta\alpha^{-1}\beta^{-1}) &= H(\alpha)H(\beta)H(\alpha^{-1})H(\beta^{-1}) = H(\alpha)H(\alpha^{-1})H(\beta)H(\beta^{-1}) = \\ &= H(\alpha)H(\alpha)^{-1}H(\beta)H(\beta)^{-1} = e \end{aligned}$$

so that $Comm(L_\star(M)) \subseteq K(P(M,G))$, this inclusion is opposed to the one in 1. and so, when G is Abelian, $Comm(L_\star(M)) = K(P(M,G))$. □

As an immediate consequence of 2. one gets that

if G is Abelian then $\mathcal{H}_\star(M,G) = L_\star(M)/Comm(L_\star(M))$

this is an Abelian group called the **Abelianized loop group**.

It follows that *the dependence of the hoop group on the gauge group G vanishes when G is Abelian.*

By a physical point of view this result implies that *every Abelian unitary gauge theory has the same hoop group*, the prototype of the Abelian hoop group is then $\mathcal{H}_\star(M, U(1))$.

The last two steps which lead to the relations between the loop groups are the following lemmas, in which plays a central role the non-Abelian Lie group $SU(2) := \{M \in SL(2,\mathbb{C}) \mid M^{-1} = M^\dagger\}$, where $M^\dagger \equiv {}^t\overline{M}$.

Lemma 2.3.1 *$SU(2)$ admits a non-Abelian free subgroup generated by two elements.*

The proof of this fact can be found in [1].

Lemma 2.3.2 *If G admits a subgroup isomorphic to $SU(2)$ then the holonomic kernel of every principal fiber bundle which has G as structure group is trivial, i.e. $K(P(M,G)) = \{\star\}$.*

Proof. By the preceding lemma we know that there exists a non-Abelian free subgroup of G contained in $SU(2) \subseteq G$, generated by two elements, say g and h.

Take a loop $\alpha \in L_\star(M)$ and factorize it as the product of powers of a simple independent family $\{\beta_i\} \subset L_\star(M)$ in the usual way:

$$\alpha = {\beta_1}^{n_{1,1}} \cdots {\beta_m}^{n_{m,1}} {\beta_1}^{n_{1,2}} \cdots {\beta_m}^{n_{m,k}} .$$

Then take a connection A such that: $H_A(\beta_1) = g$, $H_A(\beta_2) = h$ and $H_A(\beta_i) = e$ for every $i = 3, \dots, m$.

α belongs to the holonomic kernel if and only if the equation $H_A(\alpha) = e$ holds, but this equation gives a defining relation for the subgroup of $SU(2)$ generated by g and h, thus $n_{1,j} = n_{2,j} = 0 \ \forall j$.

The same argument works for all the others factors and this implies that $\alpha = \star$. □

Now the theorem that describes the relations between the loop groups can finally be stated and proved:

Theorem 2.3.5 *The following assertions hold:*

1. *if G contains a subgroup isomorphic to $SU(2)$ then $\mathcal{H}_\star(M,G)$ is isomorphic to $L_\star(M)$;*

2. *$L_\star(M)$ and $\mathcal{L}_\star(M)$ are always isomorphic.*

Proof. 1. $\mathcal{H}_\star(M,G) = L_\star(M)/K(P(M,G))$, but $K(M,G) = \{\star\}$ when G satisfies the conditions in 1. and so $L_\star(M) \simeq \mathcal{H}_\star(M,G)$.

2. It has already been proved that $\mathcal{H}_\star(M,G) \quad \mathcal{L}_\star(M) \quad L_\star(M)$ and that $\mathcal{H}_\star(M,G)$ and $L_\star(M)$ agrees when $SU(2) \subseteq G$, but $\mathcal{L}_\star(M)$ doesn't depend on G, hence it agrees with $L_\star(M)$ for every choice of G. □

I stress that the theorem contemplates the gauge theories with $G = SU(N)$ (and so the Yang-Mills theories), but not the case $G = U(1)$ (or, generally, G Abelian) and in fact $L_\star(M) \neq \mathcal{H}_\star(M, U(1))$ because the last group is Abelian, while the first and the second aren't.

Chapter 3

The role of the holonomy in gauge theory

In this chapter is investigated the equivalence between gauge-equivalence classes of connections and conjugation classes of holonomy maps, this equivalence will show the great importance of the holonomies in the modern formulation of gauge theories.

3.1 Kobayashi's representation theorem

The representation theorem deals with this problem: is it possible to know the connection which induces a holonomy map by knowing only the action of this map on the group of loops? The answer is negative for a single connection, what is true is that there is a one-to-one correspondence between holonomies and the set of connections modulo some special kinds of gauge transformations. More important, the conjugation classes of holonomy maps are in one to one correspondence with the gauge equivalence classes of connections, as proved for the first time by Kobayashi [25] in 1954.

To make this assertions precise let us introduce the normal subgroup $\mathcal{G}_\star$ of $\mathcal{G}$ given by the gauge transformations on P which act as the identity on the fiber over the point $\star \in M$.

The first result which leads to Kobayashi's reconstruction theorem is the following.

Lemma 3.1.1 *Let $P \equiv P(M,G)$ and $P' \equiv P'(M,G)$ be two PFB with the same base and gauge group but not necessary equal total spaces. Let also A and A' be two connections of P and P', respectively. Finally fix a point $\star \in M$ and two arbitrary points $p_0 \in P_\star$, $p'_0 \in P'_\star$.*

Then $H_A = H_{A'}$ if and only if there exists a G-equivariant isomorphism $\varphi : P \to P'$ which induces the identity on M and such that:

$$\begin{cases} \varphi^* A' = A \\ \varphi(p_0) = p'_0 \, . \end{cases}$$

Proof.

$\Leftarrow$: suppose that there exists an isomorphism φ with the required properties, then it would map A-horizontal paths in P into A'-horizontal paths in P' making the following diagram:

$$\begin{array}{ccc} P'_\star & \xrightarrow{\wp_{\alpha,A'}} & P'_\star \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ P_\star & \xrightarrow[\wp_{\alpha,A}]{} & P_\star \end{array}$$

commutative, i.e. $\varphi(\wp_{\alpha,A}(p_0)) = \wp_{\alpha,A'}(\varphi(p_0))$ for every fixed point p_0 in $P_\star$.

Moreover, by definition of parallel transport and thanks to the equivariance of φ one would have $\varphi(\wp_{\alpha,A}(p_0)) = \varphi(p_0.H_A(\alpha)^{-1}) = \varphi(p_0).H_A(\alpha)^{-1} = p'_0.H_A(\alpha)^{-1}$.

Furthermore $\wp_{\alpha,A'}(\varphi(p_0)) = \wp_{\alpha,A'}(p'_0) = p'_0.H_{A'}(\alpha)^{-1}$.

The commutativity of the diagram implies that $p'_0.H_A(\alpha)^{-1} = p'_0.H_{A'}(\alpha)^{-1}$, and so, because of the freedom of the action, $H_{A'}(\alpha) = H_A(\alpha)$, for every $\alpha \in L_\star(M)$, hence $H_A = H_{A'}$.

$\Rightarrow$: suppose now that $H_A = H_{A'}$, then one has to show that it exists a G-equivariant isomorphism with the properties listed in the thesis.

First of all observe that, for every $g \in G$, the map

$$\begin{array}{rccc} \varphi_\star : & P_\star & \longrightarrow & P'_\star \\ & p_0.g & \mapsto & \varphi_\star(p_0.g) := p'_0.g \end{array}$$

is a G-equivariant diffeomorphism, thanks to the fact that the action of G is free and transitive on the fibers.

Remember also that, fixed an arbitrary path γ starting in $\star$ and ending in any other fixed point $x \in M$, the parallel transport along γ associated to every connection is also a G-equivariant diffeomorphism, hence the map $\varphi_x : P_x \to P'_x$, $\varphi_x := \wp_{\gamma,A'} \circ \varphi_\star \circ \wp_{\gamma,A}{}^{-1}$ is the composition of G-equivariant diffeomorphisms and so it is a G-equivariant diffeomorphism itself.

The construction is resumed by the following commutative diagram

$$\begin{array}{ccc} P_x & \xrightarrow{\varphi_x} & P'_x \\ {\scriptstyle \wp_{\gamma,A}} \uparrow & & \uparrow {\scriptstyle \wp_{\gamma,A'}} \\ P_\star & \xrightarrow[\varphi_\star]{} & P'_\star \,. \end{array}$$

φ_x is independent from the choice of the path γ, in fact the contribution of γ is contained only in the parallel transports $\wp_{\gamma,A'}$ and ${\wp_{\gamma,A}}^{-1}$ but these produces two elements in G which are one the inverse of the other, because we are working in the hypothesis that $H_A = H_{A'}$.

It follows that the map $\varphi : P \to P$ defined by $\varphi|_{P_x} := \varphi_x$, $\forall x \in M$, is a G-equivariant isomorphism $\varphi : P \to P$ which projects on the identity of M through the relation 1.1, this follows immediately from the fact that every φ_x has domain and range in a fiber over the same point x (although in different total spaces).

By construction $\varphi \circ \wp_{\alpha,A} = \wp_{\alpha,A'} \circ \varphi$ and, thanks to the properties of the parallel transport, $\varphi^* A' = A$.

Hence φ has all the properties required in the thesis of the theorem. □

If, in particular, one chooses $P' = P$, then one has to fix only the point $p_0 \in P_\star$ and so φ acts as the identity on the fiber $P_\star$. Hence the lemma just proven asserts that $H_A = H_{A'}$ if and only if A and A' are related each other by a gauge transformation $\varphi \in \mathcal{G}_\star$, hence:

$$\mathcal{A}/\mathcal{G}_\star \simeq Hom_P(L_\star(M), G),$$

where $Hom_P(L_\star(M), G)$ denotes the subset of $Hom(L_\star(M), G)$ given by the holonomy maps, thus the orbits of $\mathcal{A}$ w.r.t. $\mathcal{G}_\star$ are faithfully represented by their corresponding holonomy maps.

As an immediate corollary one has the following result.

Corollary 3.1.1 *A choice of a trivialization in a given point $\star \in M$ enables the identification of a connection with its holonomy map along a loop based in $\star$; analogously, a trivialization in the starting and the ending point of a path in M enables the identification of a connection with its parallel transport along the path.*

The previous result can be refined as stated below.

Lemma 3.1.2 *Let the hypothesis of the previous lemma be satisfied, then it exists a G-equivariant isomorphism $\varphi : P \to P'$ inducing the identity on M and satisfying $\varphi^* A' = A$ if and only if both the following conditions are satisfied:*

1. *there exists a G-equivariant map $\varphi_\star : P_\star \to P'_\star$;*

2. $H_{A'}(\alpha) = Ad_{g^{-1}} H_A(\alpha)$, $\forall \alpha \in L_\star(M)$, *where g is the only element of G such that:* $p'_0 = \varphi(p_0).g$.

Proof.

$\Rightarrow$: first of all observe that, by definition of holonomy, $\wp_{\alpha,A}(p_0) = p_0.H_A(\alpha)^{-1}$, hence $\wp_{\alpha,A}(p_0).H_A(\alpha) = p_0.H_A(\alpha)^{-1}H_A(\alpha) = p_0$. Suppose now that there exists a map φ with the required properties, then $\varphi_\star$ can be defined simply as the restriction of it to the fiber $P_\star$ and

$$p'_0 := \varphi(p_0).g = \varphi(\wp_{\alpha,A}(p_0).H_A(\alpha)).g = \varphi(\wp_{\alpha,A}(p_0)).H_A(\alpha)g$$

but, as in the proof of the previous lemma, one finds that $\varphi(\wp_{\alpha,A}(p_0)) = \wp_{\alpha,A'}(\varphi(p_0))$, hence

$$\begin{aligned} p'_0 &= \wp_{\alpha,A'}(\varphi(p_0)).H_A(\alpha)g = \wp_{\alpha,A'}(p'_0.g^{-1}).H_A(\alpha)g = \\ &= \wp_{\alpha,A'}(p'_0).g^{-1}H_A(\alpha)g = p'_0.H_{A'}(\alpha)^{-1}g^{-1}H_A(\alpha)g\,. \end{aligned}$$

Writing $p'_0 = p'_0.e$ one has: $p'_0.e = p'_0.H_{A'}(\alpha)^{-1}g^{-1}H_A(\alpha)g$ it follows that $e = H_{A'}(\alpha)^{-1}g^{-1}H_A(\alpha)g$, but then $H_{A'}(\alpha) = g^{-1}H_A(\alpha)g$ for every loop.

$\Leftarrow$: suppose that there exists a G-equivariant map $\varphi_\star : P_\star \to P'_\star$ and that $H_{A'}(\alpha) = g^{-1}H_A(\alpha)g$ for every loop, then, for any path γ such that $\gamma(0) = \star$ and $\gamma(1) = x$, define $\varphi_x : P_x \to P'_x$, $\varphi_x = \wp_{\gamma,A'} \circ \varphi_\star \circ \wp_{\gamma,A}{}^{-1}$. Following the same arguments of the proof of the previous lemma one easily obtains the thesis. □

As a particular case of the last lemma one can take $P \equiv P'$, then φ becomes a gauge transformation Φ and the next, important, theorem follows immediately.

Theorem 3.1.1 (Representation theorem) *Let A and A' be two connections on a principal fiber bundle $P(M,G)$, then it exists a gauge transformation Φ of P such that*

$$A = \Phi^* A' \qquad \textit{(i.e. } A \textit{ and } A' \textit{ are gauge-equivalent connections)}$$

if and only if there exists $g \in G$ such that $H_{A'}(\alpha) = Ad_{g^{-1}}H_A(\alpha)$, $\forall\, \alpha \in L_\star(M)$.

The representation theorem can be synthetically symbolized as:

$$\mathcal{A}/\mathcal{G} \simeq Hom_P(L_\star(M), G)/Ad_G\,.$$

As discussed in chapter 1., $\mathcal{A}/\mathcal{G}$ is the configuration space of the gauge theories, thus the representation theorem shows the remarkable fact that

the physically distinct configurations of the classical gauge theories are in one-to-one correspondence with the conjugation classes of holonomy maps.

3.2 The reconstruction theorem

The reconstruction theorem deals with the problem to find under what conditions a homomorphism of $Hom(L_\star(M), G)$ is characterizable as the holonomy map associated to a certain connection A.

Since the connections here are always assumed to be smooth w.r.t. a fixed differential structure, it is plausible to suppose that the reconstruction of the holonomy map from a homomorphism of $Hom(L_\star(M), G)$ is possible only after the introduction of a convenient differential structure on $L_\star(M)$. This is indeed how things go.

A first important observation is this: $L_\star(M)$ **is not a Lie group**, in fact in [18] it has been shown that $L_\star(M)$ doesn't contain any smooth 1-parameter subgroup.

However it is possible to give $L_\star(M)$ a smooth differential structure generated by the set of paths in $L_\star(M)$ given by:

$$\{\gamma : \mathbb{R} \to L_\star(M) \ \mid \ \gamma(t) = \alpha_t\}$$

where α_t is defined through:

$$h : \mathbb{R} \times [0, 1] \to M, \quad h(t, s) := \alpha_t(s)$$

and h is required to be a continuous map such that, for at least a partition $0 = s_1 < s_2 < \ldots < s_k = 1$, $h : \mathbb{R} \times (s_{i-1}, s_i) \to M$ is globally smooth and analytic when the first variable is fixed, for every $i = 1, \ldots , k$.

Def. 3.2.1 *A homomorphism $H : L_\star(M) \to G$ is said to be smooth if the path $\mathbb{R} \ni t \mapsto H(\alpha_t) \in G$ is smooth for every smooth curve $\mathbb{R} \ni t \mapsto \alpha_t \in L_\star(M)$.*

The set of all homomorphism $H : L_\star(M) \to G$ smooth in this sense will be indicated by $Hom^\infty(L_\star(M), G)$.

This definition fits our purposes because the holonomies born as solutions of differential equations in G generated by vector fields smoothly depending on a real parameter.

In [27] Lewandowski proved the important:

Theorem 3.2.1 (Reconstruction theorem) *The correspondence*

$$\begin{array}{rccl} H: & \mathcal{A}/\mathcal{G} & \longrightarrow & Hom^{\infty}(L_\star(M), G)/Ad_G \\ & [A]_{\mathcal{G}} & \mapsto & H([A]_{\mathcal{G}}) := [H_A]_{Ad_G} \end{array}$$

is a bijection.

The proof of this theorem is quite technical and doesn't involve arguments of interest for the later purposes, hence it is omitted.

Here it is important to observe that, since gauge-equivalence classes of connections are equivalent to Ad_G-equivalence classes of holonomy maps (by virtue of the representation theorem), the reconstruction theorem implies that

Ad_G-equivalence classes of holonomy maps are in one to one correspondence with Ad_G-equivalence classes of smooth homomorphisms from $L_\star(M)$ to G.

It is interesting to analyze what happens if one requires the connections A to be only $\mathcal{C}^k$, $k \geq 0$, instead of smooth: again in [27] it is proved that under this condition the holonomy map is a $\mathcal{C}^{k+1}$-homomorphism in the sense that, for every smooth path $\mathbb{R} \ni t \mapsto \alpha_t \in L_\star(M)$, the curve $\mathbb{R} \ni t \mapsto H(\alpha_t) \in G$, is $\mathcal{C}^{k+1}$.

At the end of this process one gets algebraic homomorphisms $H : L_\star(M) \to G$ to which do not correspond any connection in the ordinary sense, but objects called **generalized connections**.

The loop groups are topological groups in the topology induced from the $\mathcal{C}^\infty$ curves, called c^∞ **topology**.

A base of open neighborhoods of the unit loop $\star$ in this topology consists of subsets W of $L_\star(M)$ such that for every $\mathcal{C}^\infty$ curve $\mathbb{R} \ni t \mapsto \alpha_t \in L_\star(M)$, the set $\{t \in \mathbb{R} \mid \alpha_t \in W\}$ is an open neighborhood of 0 in $\mathbb{R}$.

Every $\mathcal{C}^\infty$ curve is c^∞ continuous.

Theorem 3.2.2 *$L_\star(M)$ is a c^∞ topological Hausdorff group.*

Proof. It is sufficient to observe that if α_t and β_t are homotopies of $\mathcal{C}^k$ loops then $\gamma_t := \alpha_t\beta_t$ and $\delta_t := {\alpha_t}^{-1}$ are also homotopies of $\mathcal{C}^k$ loops, thus the group operations are smooth in the c^∞ topology.

To prove that $L_\star(M)$ is a Hausdorff group suppose $\alpha \neq \star$, choose any PFB with base M and gauge group $SU(2)$ and consider a smooth connection A such that $H_A(\alpha) \neq e$. Since H_A is a $\mathcal{C}^\infty$ map, it is c^∞ continuous.

By using H_A one can construct an open neighborhood of $\star$ not containing α and this implies the Hausdorff separation property. □

Chapter 4

The Wilson functions, the holonomy C^*-algebra and its spectrum

4.1 The Wilson functions

A step of paramount importance for the development of the loop quantization of gauge theories is the recognition of a separating set of gauge-invariant functions of connections.

If the gauge group of a gauge theory is $U(N)$ or $SU(N)$, this functions happen to be the Wilson functions, also known in lattice gauge theory as **Wilson's loop**. The reason of this name relies in the fact that every Wilson function T_α is labelled by a loop α in M and it is defined to be the complex valued function on $\mathcal{A}/\mathcal{G}$ which maps a gauge-equivalence class of connections $[A]$ into the normalized trace of the holonomy $H_A(\alpha)$, where A is any representative of $[A]$, i.e.

$$\begin{array}{llll} T_\alpha : & \mathcal{A}/\mathcal{G} & \longrightarrow & \mathbb{C} \\ & [A] & \mapsto & T_\alpha([A]) := \frac{1}{N} Tr(H_A(\alpha)) \end{array}$$

where Tr means the trace operator taken in the fundamental representation of the gauge group G.

The definition of T_α is well posed because, if A and A' are two gauge-equivalent connections, then the representation theorem assures that there exist an element $g \in G$ such that $H_{A'}(\alpha) = g^{-1} H_A(\alpha) g$, but, thanks to the cyclic property of the trace, one gets $Tr(H_{A'}(\alpha)) = Tr(g^{-1}H_A(\alpha)g) = Tr(gg^{-1}H_A(\alpha)) = Tr(H_A(\alpha))$. This property is summarized by saying that **the Wilson functions are gauge-invariant**.

Since the Wilson functions are labelled by loops, it is worth remembering that we are dealing with piecewise analytic loops. The regularity of the loops has many remarkable consequences in the development of the loop quantization. The analysis of the piecewise smooth regularity is still under investigation.

Notice that the definition is well posed for loops α belonging to $L_\star(M) \equiv \mathcal{L}_\star(M)$ and also for loops α belonging to $\mathcal{H}_\star(M,G)$, in fact:

- suppose $\alpha \sim_{el} \beta$ then there exists an immediately retraced loop $\gamma = \prod_i \gamma_i {\gamma_i}^{-1}$ such that $\alpha = \beta\gamma$, applying H_A and using his factorization property one has $H_A(\alpha) = H_A(\beta) \prod_i H_A(\gamma_i) {H_A(\gamma_i)}^{-1} = H_A(\beta)$, hence $T_\alpha = T_\beta$;
- suppose now $\alpha \sim_{hol} \beta$, then, by definition, $H_A(\alpha) = H_A(\beta)$ for every $A \in \mathcal{A}$, thus $T_\alpha = T_\beta$.

Observe also that, since the holonomy maps are unitary representations of the loop group, the Wilson functions induce, by duality, the maps

$$\begin{array}{rccl} T^{[A]}: & L_\star(M) & \longrightarrow & \mathbb{C} \\ & \alpha & \mapsto & T^{[A]}(\alpha) := T_\alpha([A]) \end{array}$$

which are the normalized characters of the unitary representations of the loop group given by the holonomy maps.

The Wilson functions belong to $\mathcal{C}_b(\mathcal{A}/\mathcal{G})$, the continuity follows obviously from the continuity of the trace and the boundness follows from the fact that Tr is invariant under the choice of the base w.r.t. the matrix $H_A(\alpha)$ is represented, thus one can choose the base in which $H_A(\alpha)$ is diagonal, in this case $Tr(H_A(\alpha))$ is the sum of the eigenvalues of the matrix, which are all bounded in absolute value by 1 thanks to the fact that, for any $n \times n$ unitary matrix U, one has $U^\dagger U = I$, i.e. $\sum_{k=1}^n \overline{u_{ik}} u_{jk} = \delta_{ij}$, hence $|Tr(H_A(\alpha))| \leq N$ and

$$\|T_\alpha(A)\|_\infty := \sup_{[A]\in\mathcal{A}/\mathcal{G}} |T_\alpha([A])| \leq 1$$

thanks to the normalization factor.

The Wilson functions satisfy the so-called **Mandelstam identities** of the first and the second kind.

The Mandelstam identities of the first kind hold for every gauge group and they are a simple consequence of the cyclic property of the trace:

$$T_{\alpha\beta} = T_{\beta\alpha}$$

for every couple of loops α and β.

The proof is very easy: by using the factorization property of the holonomy maps and the already mentioned cyclic property of the trace one has, for every $[A] \in \mathcal{A}/\mathcal{G}$, one has:

$$\begin{aligned} T_{\alpha\beta}([A]) &= \tfrac{1}{N}Tr(H_A(\alpha\beta)) = \tfrac{1}{N}Tr(H_A(\alpha)H_A(\beta)) = \\ &= \tfrac{1}{N}Tr(H_A(\alpha)H_A(\beta)H_A(\beta)^{-1}H_A(\beta)) = \\ &= \tfrac{1}{N}Tr(H_A(\beta)H_A(\alpha)H_A(\beta)H_A(\beta)^{-1}) = \\ &= \tfrac{1}{N}Tr(H_A(\beta)H_A(\alpha)) = \tfrac{1}{N}Tr(H_A(\beta\alpha)) = \\ &= T_{\beta\alpha}([A])\,. \end{aligned}$$

An immediate consequence of the Mandelstam identities of the first kind is that **the Wilson functions are not independent**, in fact, although the loops $\gamma := \alpha\beta$ and $\eta := \beta\alpha$ are different, the Wilson functions they induce are the same.

To discuss the Mandelstam identities of the second kind one has to distinguish between the monodimensional case, when $G = U(1)$, and the other situations.

When $G = U(1)$ the Mandelstam identities of the second kind simply reflect the property that the trace reduces to the identity operator, thus the factorization property of the holonomy maps extends to the Wilson functions:

$$T_{\alpha\beta} = T_\alpha T_\beta \qquad \text{if } G \equiv U(1)$$

for every loop α and β.

When $N > 1$, the Mandelstam identities of the second kind instead follows from combinatorial arguments strongly depending on the group structure, these arguments are quite technical and not very useful for the later purposes, the only important thing to mention here is that the Mandelstam identities of the second kind relative to any subgroup of $SL(N, \mathbb{C})$ allow to write down the product of the traces of N special matrices as a linear combination of the traces of $N-1$ special matrices, see [18] or [19] for the proof.

In particular this result implies that *the product of the traces of a finite number of* 2×2 *special matrices can be written as a linear combination of the traces of these matrices.*

The immediate consequence is that, *when* G *is a subgroup of* $SL(2, \mathbb{C})$, *the algebra generated by the Wilson functions agrees with their linear span.*

As an example the Mandelstam identities of the second kind for $SU(2)$ are:

$$T_\alpha T_\beta = \frac{1}{2}(T_{\alpha\beta} + T_{\alpha\beta^{-1}}) \qquad \text{if } G \equiv SU(2)$$

for every loop α and β, as one can easily verify by direct computation using the Cayley-Klein parameterization of a generic $SU(2)$-matrix, i.e.

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

where a and b are complex numbers such that $|a|^2 + |b|^2 = 1$.

This parameterization also shows that, **when $G = SU(2)$, the Wilson functions are real-valued**, in fact the normalized trace of the generic $SU(2)$-matrix above is precisely $\Re(a)$.

To conclude this section dedicated to the Wilson functions it is worth remembering that these ones has been introduced in the mathematical-physics literature in 1974 by Kenneth Wilson in a pioneering article [43] in which he used the traces of the holonomy to construct a manifestly gauge-invariant action of a hyper-cubic lattice gauge theory in the Minkowski space-time exhibiting the confinement of a static couple of quark and anti-quark.

The *confinement of quarks* is the name used by physicists to describe the phenomenon that the quarks never show themselves in a free state, but they are always observed mixed in the adrons, i.e. composites of quarks with null color charge.

Wilson studied a couple of quark and antiquark created at the instant $t = 0$ at distance R and annihilating at the instant $t = \tau$.

The situation is described, on a hypercubic lattice in Minkowski space-time, by the so-called *Wilson action*:

$$S := \frac{1}{N{g_0}^2} \sum_{\alpha} S_\alpha$$

where α is a rectangle in the lattice of vertexes i, j, k, l and where:

$$S_\alpha := -tr(U_{ij} U_{jk} U_{kl} U_{li})$$

being g_0 the coupling constant of the interaction quark-antiquark and U_{ij} the parallel transport matrix (w.r.t. a fixed gauge potential A^a_μ) relative to the segment starting in i and ending in j (analogous for the other matrices).

The *vacuum expectation value* of the Wilson functions in this theory is given by:

$$< T_\alpha > := \frac{\int \frac{1}{N} tr(\prod_\alpha U_{kl}) \, e^{-S} \prod_\alpha d\mu_H}{\int e^{-S} \prod_\alpha d\mu_H}$$

where:

- $\prod_\alpha U_{kl}$ symbolizes the ordered product of the matrices associated to the segments composing the loop α in the lattice (*the discrete version of the path-ordered exponential*);

- $\prod_\alpha d\mu_H$ represents the product Haar measure (a Haar measure for every segment which appear in the loop α).

The computation of the binding energy $E(R)$ of the static couple of quark and antiquark in the strong-coupling approximation ($g_0 \to \infty$) and for long times ($\tau \to \infty$) within this model is the following:

$$E(R) \sim Rlog({g_0}^2).$$

This formula shows that the confining potential energy of the couple grows linearly with the distance R between the particles thus prohibiting their macroscopic disjoining.

To get more information about the development of this important topic the interest reader is referred to [35] and [45].

4.2 The overcompleteness of the Wilson functions

In this section it will be proved the overcompleteness of the Wilson functions by using the abstract theory of topological groups and their representations, this proof strongly depends on the fact that the gauge group is chosen to be a unitary (or special unitary) matrix group.

In literature one speaks of overcompleteness to describe the fact that the Wilson functions can separate the classical degrees of freedom of the gauge theories, i.e. the elements of $\mathcal{A}/\mathcal{G}$, but they are not independent.

Since the Wilson functions induce the normalized characters $T^{[A]}$ of the representations of the loops groups given by the holonomy maps, if the loop group was compact then the completeness of the Wilson functions would be immediate since, for compact groups, the characters are in bijection with the equivalence classes of representations.

However the loop group is a rather complicated non-compact group, hence the proof of the completeness of Wilson functions is a non-trivial result.

Let G be an arbitrary group and let $U : \mathsf{G} \to \mathcal{U}(\mathcal{H})$ denote a unitary representation of G into the group $\mathcal{U}(\mathcal{H})$ of the unitary operators on a Hilbert space $\mathcal{H}$ of finite dimension N.

To every such representation U one can associate its **normalized character**, i.e. the map:

$$\begin{array}{rccl} \tau_U : & \mathsf{G} & \longrightarrow & \mathbb{C} \\ & g & \mapsto & \tau_U(g) := \frac{1}{N} Tr(U(g)). \end{array}$$

Two representations $U, \tilde{U}$ of G are said to be **equivalent** if there exists an *invertible* intertwining operator A on $\mathcal{H}$ between them, i.e. an invertible operator A such that $\tilde{U}(g) = AU(g)A^{-1}$, for every $g \in \mathsf{G}$.

From the cyclic property of the trace it follows that two equivalent representations $U, \tilde{U}$ have the same normalized character: $\tau_U = \tau_{\tilde{U}}$.

An equivalence class of unitary representations of G will be indicated by λ, one of its representative by U^λ and the normalized character of one of its representatives by τ_λ.

It is an important fact that, for unitary representations, equivalence classes and unitary equivalence classes agree, as stated in the following theorem.

Theorem 4.2.1 *Two unitary representations U and $\tilde{U}$ of G belong to the same equivalence class if and only if they belong to the same unitary equivalence class.*

Proof.

$\Leftarrow$: U and $\tilde{U}$ belong to the same unitary equivalence class if there exists an *unitary* intertwining operator A such that $\tilde{U}(g) = AU(g)A^{-1}$, for every $g \in \mathsf{G}$. The unitary operators are invertible, hence the implication $\Leftarrow$ is obvious.

$\Rightarrow$: now assume U and $\tilde{U}$ to be equivalent representations of G with intertwining operator given by the *invertible* operator A.

By taking the adjoint of both sides of the intertwining relation one has: $\tilde{U}(g)^\dagger = (A^{-1})^\dagger U(g)^\dagger A^\dagger = (A^\dagger)^{-1} U(g)^\dagger A^\dagger$.

Since $U(g)$ and $\tilde{U}(g)$ are unitary operators they satisfy the identities $U(g)^\dagger = U(g)^{-1}$ and $\tilde{U}(g)^\dagger = \tilde{U}(g)^{-1}$ thus the previous relation can be written as: $\tilde{U}(g)^{-1} = (A^\dagger)^{-1} U(g)^{-1} A^\dagger$.

By taking the inverse of both sides one obtains $\tilde{U}(g) = (A^\dagger)^{-1} U(g) A^\dagger$, which, conjugated by $AA^\dagger$, transforms into $AA^\dagger \tilde{U}(g)(AA^\dagger)^{-1} = AU(g)A^{-1} = \tilde{U}(g)$ so that $\tilde{U}(g)$ commutes with $AA^\dagger$.

Thanks to the spectral theorem this also implies that $|A| := \sqrt{AA^\dagger}$ commutes with $\tilde{U}(g)$, hence the polar decomposition of the invertible operator A can be written as: $A = |A|B$, where B is a unitary operator.

By substituting this expression of A in the intertwining relation one gets $\tilde{U}(g) = |A|BU(g)(|A|B)^{-1}$, i.e. $\tilde{U}(g) = |A|BU(g)B^{-1}|A|^{-1}$, that is $|A|^{-1}\tilde{U}(g)|A| = BU(g)B^{-1}$. But $[\tilde{U}(g), |A|] = 0$ hence $|A|^{-1}\tilde{U}(g)|A| = |A|^{-1}|A|\tilde{U}(g) = \tilde{U}(g)$.

Since B is unitary, the relation $\tilde{U}(g) = BU(g)B^{-1}$ proves that $U(g)$ and $\tilde{U}(g)$ are unitary equivalent. □

Now it is worth introducing the concept of the compact group associated to any topological group: let $\mathfrak{G}$ be a topological group and let $\mathcal{B}(\mathfrak{G})$ be the space of the bounded complex-valued functions on $\mathfrak{G}$.

The left translation by the element $g \in \mathfrak{G}$ defines an action of $\mathfrak{G}$ on $\mathcal{B}(\mathfrak{G})$ given by:

$$\begin{array}{ccc} \mathfrak{G} \times \mathcal{B}(\mathfrak{G}) & \longrightarrow & \mathcal{B}(\mathfrak{G}) \\ (g, f) & \mapsto & L_g f \end{array}$$

where $L_g f : \mathfrak{G} \to \mathbb{C}$, $L_g f(h) := f(g^{-1}h)$, for every $h \in \mathfrak{G}$.

This action is also well defined, by restriction, on $\mathcal{C}_b(\mathfrak{G})$, the space of bounded continuous complex-valued functions on $\mathfrak{G}$.

$\mathcal{B}(\mathfrak{G})$ and $\mathcal{C}_b(\mathfrak{G})$ are Abelian C^*-algebras w.r.t. the $\| \ \|_\infty$-norm and the correspondence $f \to L_g f$ is an isometry for both of them.

Def. 4.2.1 *The* **left orbit** *of a function $f \in \mathcal{B}(\mathfrak{G})$ is the closure of the set $\{L_g f \mid g \in \mathfrak{G}\}$ w.r.t. the $\| \ \|_\infty$-norm. It is denoted by X_f.*

If X_f is a compact set then f is said to be **almost periodic***. The set of all almost periodic functions on $\mathfrak{G}$ is denoted by $A(\mathfrak{G})$.*

Standard examples of almost periodic functions are the characters of the continuous representations of a compact group.

It is obvious that the left translations carry $A(\mathfrak{G})$ into itself and acts isometrically on every orbit X_f, hence the closure of the group of the left translations in the topology of the punctual convergence, denoted by $\chi(\mathfrak{G})$, is a closed subgroup of $\prod_{f \in A(\mathfrak{G})} Iso(X_f)$, where $Iso(X_f)$ is the group of the isometries of X_f, which is a compact Hausdorff group in the topology of punctual convergence by a standard theorem of topology.

It follows that also $\chi(\mathfrak{G})$ is a compact Hausdorff group (w.r.t. the induced topology) and it is called **the compact group associated to the topological group** $\mathfrak{G}$.

It can be proved that the map

$$\begin{array}{cccc} \chi : & \mathfrak{G} & \longrightarrow & \chi(\mathfrak{G}) \\ & g & \mapsto & \chi(g) \end{array}$$

where $\chi(g) := \{L_g|_{X_f} \ , \ f \in A(\mathfrak{G})\}$, is a continuous homomorphism with dense range.

Furthermore $\chi(\mathfrak{G})$ has the following universal property ([24]):

Theorem 4.2.2 *If $\Phi : \mathfrak{G} \to K$ is a continuous homomorphism from the topological group $\mathfrak{G}$ into the compact group K, then it always exists a continuous homomorphism $\varphi : \chi(\mathfrak{G}) \to K$ such that the following diagram*

$$\begin{array}{ccc} \chi(\mathfrak{G}) & \xrightarrow{\varphi} & K \\ {\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle\Phi} \\ \mathfrak{G} & = & \mathfrak{G} \end{array}$$

commutes, i.e. $\varphi \circ \chi = \Phi$.

This results allows to extend a well known theorem involving unitary representations of compact groups to unitary representations of every topological groups.

Theorem 4.2.3 *Let $\mathfrak{G}$ be an arbitrary topological group. Let also U and $\tilde{U}$ be two continuous unitary representations of $\mathfrak{G}$ with normalized characters τ_U and $\tau_{\tilde{U}}$. If $\tau_U = \tau_{\tilde{U}}$, then U and $\tilde{U}$ belong to the same unitary equivalence class.*

Proof. Thanks to the universal property of the compact group associated to $\mathfrak{G}$ one can extend every continuous unitary representation $U : \mathfrak{G} \to U(N)$ of $\mathfrak{G}$ to the unique continuous unitary representation $V : \chi(\mathfrak{G}) \to U(N)$ of $\chi(\mathfrak{G})$ that makes the following diagram

$$\begin{array}{ccc} \chi(\mathfrak{G}) & \xrightarrow{V} & U(N) \\ {\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle U} \\ \mathfrak{G} & = & \mathfrak{G} \end{array}$$

commute, i.e. $U = V \circ \chi$. The same considerations holds for $\tilde{U}$ with analogous notations.

Let T_V be the normalized character of V, then $(\frac{1}{N}Tr) \circ U = (\frac{1}{N}Tr) \circ V \circ \chi$, i.e. $\tau_U = \mathsf{T}_V \circ \chi$ and the same thing holds for the normalized character $\mathsf{T}_{\tilde{V}}$ of $\tilde{V}$.

Thanks to the fact that $\mathfrak{G}$ is dense in $\chi(\mathfrak{G})$, $\tau_U = \tau_{\tilde{U}}$ implies $\mathsf{T}_V = \mathsf{T}_{\tilde{V}}$, but these are the normalized character of continuous representations of compact groups, hence V and $\tilde{V}$ are equivalent for a well known result (see for instance [13] or [39]).

Thanks to the commutativity of the last diagram, this implies that also U and $\tilde{U}$ are equivalent, hence, by theorem 4.2.1, it follows that V and $\tilde{V}$ belong to the same unitary equivalence class. □

The previous theorem can be generalized to *every* group G.

Corollary 4.2.1 *Let* G *be an arbitrary group. Let also* $U : \mathsf{G} \to U(N)$ *and* $\tilde{U} : \mathsf{G} \to U(N)$ *be two unitary representations with normalized characters* τ_U *and* $\tau_{\tilde{U}}$. *If* $\tau_U = \tau_{\tilde{U}}$, *then* U *and* $\tilde{U}$ *belong to the same unitary equivalence class.*

Proof. Endow G with the topology induced by all the unitary representations $\mathsf{G} \to U(N)$ (i.e. the smallest topology which makes them continuous). Then G becomes a topological group to which the previous theorem applies. □

The abstract theory developed until now can be specialized to the case in which the group is one of the loop groups, its unitary representations are the holonomy maps and their normalized characters are the functions $T^{[A]}$ induced by duality from the Wilson functions.

With these choices one immediately obtains that every function $T^{[A]}$ univocally characterizes a conjugation class of holonomy maps, i.e. a point in $Hom_P(L_\star(M), G)/Ad_G$, i.e. a gauge equivalence class of connections, hence the correspondence $[A] \mapsto T^{[A]}$ is bijective.

Note, however, that the set $\{T_\alpha \mid \alpha \in L_\star(M)\}$ is bigger than the set $\{T^{[A]} \mid [A] \in \mathcal{A}/\mathcal{G}\}$, because the Mandelstam identities implies that to different loops can correspond identical Wilson functions, hence the first set contains redundant copies of the same Wilson functions.

An immediate mathematical consequence of the bijection established above is that **for unitary gauge theories, the Wilson functions are separating on** $\mathcal{A}/\mathcal{G}$, i.e. $[A_1], [A_2] \in \mathcal{A}/\mathcal{G}$, $[A_1] \neq [A_2]$ implies that there exists at least a Wilson function T_α such that $T_\alpha([A_1]) \neq T_\alpha([A_2])$.

As remarked at the begin of this section, this behavior is often summarized by saying that **the Wilson functions form an overcomplete set of gauge invariant functions**.

4.3 The holonomy C^*-algebra $Hol(M, G)$ and its spectrum $\overline{\mathcal{A}/\mathcal{G}}$

By taking all the linear combinations of finite products of Wilson functions one gets an unital Abelian algebra w.r.t. punctual multiplication and with unit element given by the Wilson function associated to the unit loop $\star \in L_\star(M)$, this is the constant function $T_\star([A]) \equiv 1$, $\forall [A] \in \mathcal{A}/\mathcal{G}$.

This is also a $*$-algebra, indicated by $hol(M, G)$, w.r.t. complex conjugation and it is easy to see that

$$T_\alpha{}^* = T_{\alpha^{-1}} \, .$$

The completion of this $*$-algebra w.r.t. the topology induced by the $\| \, \|_\infty$ norm gives rise to a unital Abelian C^*-algebra[1] called **holonomy C^*-algebra** and denoted by $Hol(M,G)$ because it depends both on M and G, but not on the principal bundle $P(M,G)$, as will be proved later. More rigorously this is the *analytic holonomy C^*-algebra* because the loops taken into account are assumed to be piecewise analytic.

It is worth remembering a few facts about C^*-algebras (see [34] for the proofs). In what follows $\mathfrak{A}$ will denote an Abelian C^*-algebra with unit u. An element $a \in \mathfrak{A}$ is said to be **self-adjoint** if $a^* = a$ and **positive** if there exists an element $b \in \mathfrak{A}$ such that $a = b^*b$; a linear functional φ on $\mathfrak{A}$ is positive if $\varphi(a) \geq 0$ for every positive element $a \in \mathfrak{A}$, such a functional is always continuous and its norm is the value assumed in the unit of $\mathfrak{A}$: $\|\varphi\| = \varphi(u)$. The positive linear functionals on $\mathfrak{A}$ of unit norm are called the **states** of $\mathfrak{A}$ and they form a compact convex subset of the dual space $\mathfrak{A}^*$.

A **character** of $\mathfrak{A}$ is a non-identically zero homomorphism φ from $\mathfrak{A}$ to $\mathbb{C}$. A character is always *continuous* and has *unit norm* so that the characters of $\mathfrak{A}$ are precisely the multiplicative states of $\mathfrak{A}$. The **spectrum** of $\mathfrak{A}$, $\sigma(\mathfrak{A})$, is the set of all its characters; endowed with the w^*-topology[2] this is a **compact Hausdorff space**.

$\mathfrak{A}$ is isometrically isomorphic to the unital Abelian C^*-algebra of continuous complex-valued functions on its spectrum by means of the **Gelfand isomorphism**:

$$\begin{array}{cccc} \hat{} : & \mathfrak{A} & \longrightarrow & \mathcal{C}(\sigma(\mathfrak{A})) \\ & a & \mapsto & \hat{a} \end{array}$$

with $\hat{a}(\varphi) := \varphi(a)$. The Gelfand isomorphism preserves the positivity.

Identifying $\mathfrak{A}$ with $\mathcal{C}(\sigma(\mathfrak{A}))$ and using the Riesz-Markov theorem one has that *there is an isomorphism between positive linear functionals on $\mathfrak{A}$ and positive regular Borel measures on $\sigma(\mathfrak{A})$*. The representation of the positive linear functional φ_μ associated to the positive regular Borel measure μ is given by:

$$\varphi_\mu(a) = \int_{\sigma(\mathfrak{A})} \hat{a} \, d\mu \, .$$

The isometric behavior of the Riesz-Markov isomorphism can be understood easily by observing that $\|\varphi_\mu\| = \varphi_\mu(\mathbf{1}) = \int_{\sigma(\mathfrak{A})} \mathbf{1} \, d\mu = \mu(\sigma(\mathfrak{A}))$, but μ is positive, hence $\mu(\sigma(\mathfrak{A})) = |\mu|(\sigma(\mathfrak{A})) = \|\mu\|$, thus $\|\varphi_\mu\| = \|\mu\|$.

[1] A unital Banach $*$-algebra such that $\|a\, a^*\|_\infty = {\|a\|_\infty}^2$, for every element a of the algebra.

[2] A sequence of characters $\{\varphi_n\}$ converges to φ in the w^*-topology if and only if $\lim_{n\to\infty} < \varphi_n, a > = < \varphi, a >$ for every $a \in \mathfrak{A}$.

It immediately follows that **the states of $\mathfrak{A}$ are in bijection with the probability measures on** $\sigma(\mathfrak{A})$.

Finally, to every positive measure μ on $\sigma(\mathfrak{A})$ (alias to every positive functional φ_μ on $\mathfrak{A}$) one can associate the so-called **GNS representation**, which is given by the correspondence $a \mapsto M_{\hat{a}}$, where $M_{\hat{a}}$ is the multiplication operator on $L^2(\sigma(\mathfrak{A}), \mu)$ defined by $M_{\hat{a}}\psi := \hat{a}\psi$, for every $a \in \mathfrak{A}$ and $\psi \in L^2(\sigma(\mathfrak{A}), \mu)$.

From a physical point of view the GNS construction relates the C^*-algebraic approach to quantum physics to the standard one based on Hilbert spaces.

All these considerations and results apply to the holonomy C^*-algebra $Hol(M, G)$, whose compact Hausdorff spectrum $\sigma(Hol(M, G))$ is usually written $\overline{\mathcal{A}/\mathcal{G}}$ for reasons that will be cleared in the next section.

Denoting by $\bar{A}$ the elements of $\overline{\mathcal{A}/\mathcal{G}}$, the Gelfand isomorphism specialized to the holonomy C^*-algebra can be written as:

$$\begin{array}{rlcl} \hat{}: & Hol(M,G) & \longrightarrow & \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}}) \\ & f & \mapsto & \hat{f}, \quad \hat{f}(\bar{A}) := \bar{A}(f). \end{array}$$

The isometric isomorphism $Hol(M, G) \simeq \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$ will be used many times in the sequel.

In particular note that the Gelfand isomorphism implies that every function $f \in Hol(M, G)$, which is defined on $\mathcal{A}/\mathcal{G}$, can be extended in a unique way to the *continuous* function $\hat{f}$ on $\overline{\mathcal{A}/\mathcal{G}}$.

4.4 The dense injection of $\mathcal{A}/\mathcal{G}$ in $\overline{\mathcal{A}/\mathcal{G}}$ and its algebraic characterization

The results presented in this section has been first discussed by Ashtekar and Lewandowski in [5], then Baumgärtel encoded their arguments in a more abstract and rigorous setting (see [11]).

Here it will be followed Baumgärtel's construction, but working directly on the concrete objects of the gauge theories instead of treating the problem abstractly.

Theorem 4.4.1 *The map*

$$\begin{array}{rlcl} T: & \mathcal{A}/\mathcal{G} \simeq Hom_P(L_\star(M), G)/Ad_G & \longrightarrow & \sigma(Hol(M,G)) \\ & [A] & \mapsto & T([A]) := T_{[A]} \end{array}$$

where $T_{[A]}$ is the linear functional on $Hol(M, G)$ defined by

$$\begin{array}{rlcl} T_{[A]}: & Hol(M,G) & \longrightarrow & \mathbb{C} \\ & f & \mapsto & T_{[A]}(f) := f([A]) \end{array}$$

is a dense injection of $\mathcal{A}/\mathcal{G}$ into $\sigma(Hol(M,G))$, the spectrum of the holonomy C^*-algebra.

Proof. The map T is well defined, in fact $Hol(M,G)$ is an algebra under pointwise multiplication, thus $T_{[A]}(fh) = (fh)([A]) = f([A])h([A]) = T_{[A]}(f)T_{[A]}(h)$, $f,h \in Hol(M,G)$. Furthermore it is evident that $T_{[A]}$ is continuous since it acts as the evaluation map, hence it is a character of $Hol(M,G)$.

T is injective: if, by absurd, $[A],[A]' \in \mathcal{A}/\mathcal{G}$, $[A] \neq [A]'$, but $T_{[A]} = T_{[A]'}$, then by definition $f([A]) = f([A]')$ $\forall f \in Hol(M,G)$, but this is absurd since the Wilson functions are separating on $\mathcal{A}/\mathcal{G}$ and so there is at least a $T_\alpha \in Hol(M,G)$ such that $T_\alpha([A]) \neq T_\alpha([A]')$.

Finally, T is dense: if, by absurd, $T(\mathcal{A}/\mathcal{G})$ wasn't dense in $\sigma(Hol(M,G))$ then, thanks to Urysohn's Lemma, there would exist a *continuous* function, say F, on $\sigma(Hol(M,G))$ which takes the value 1 on some point outside $T(\mathcal{A}/\mathcal{G})$ and vanish on the entire $T(\mathcal{A}/\mathcal{G})$. But this is absurd since $f \in \mathcal{C}(\sigma(Hol(M,G)))$ and $f+F$ would be two different extensions of same function on $\mathcal{A}/\mathcal{G}$, against the fact that the Gelfand transform is an isomorphism. □

Once established this fact one would obviously like to find a simple characterization of the spectrum of the holonomy C^*-algebra. This turns to be possible, in fact $\sigma(Hol(M,G))$ is precisely the set of *all* algebraic homomorphisms $H : L_\star(M) \to G$, i.e. the set of the generalized connections modulo Ad_G-conjugation, in the terminology introduced in 3.2 .

Theorem 4.4.2 *Fixed any character $\varphi \in \sigma(Hol(M,G))$, there is one and only one Ad_G-class of algebraic homomorphisms $[H]_\varphi \in Hom(L_\star(M),G)/Ad_G$ such that: $\varphi(T_\alpha) = \frac{1}{N}Tr([H]_\varphi(\alpha))$ $\forall \alpha \in L_\star(M)$, i.e. the map*

$$\begin{array}{ccc} \sigma(Hol(M,G)) & \longrightarrow & Hom(L_\star(M),G)/Ad_G \\ \varphi & \mapsto & [H]_\varphi, \end{array}$$

is an injection.

The proof of this theorem is quite technical, the interest reader is referred to [1].

There are no indications that the injection from the spectrum $\sigma(Hol(M,G))$ to $Hom(L_\star(M),G)/Ad_G$ established in the previous theorem is also onto, however, thanks to the interpolation property of the holonomies one can prove that the correspondence in exam is also surjective.

Theorem 4.4.3 *The interpolation condition is a sufficient condition for the surjectivity of the map $\sigma(Hol(M,G)) \hookrightarrow Hom(L_\star(M),G)/Ad_G$, which is actually a bijection.*

Proof. The only thing to prove is that, if the interpolation property holds, then to every element of $Hom(L_\star(M),G)/Ad_G$ there correspond a character on $Hol(M,G)$.

First of all remember that every element of the Wilson algebra $hol(M,G)$ is a finite sum of the form

$$p = p_0 + \sum_{j_1} p_{j_1} T_{\alpha_{j_1}} + \sum_{j_1,j_2} p_{j_1,j_2} T_{\alpha_{j_1}} T\alpha_{j_2} + \dots$$

which is called a 'Wilson polynomial'.

For any given representative H of $[H] \in Hom(L_\star(M),G)/Ad_G$, define a functional φ_H on the set of Wilson functions as:

$$\varphi_H(T_\alpha) := \frac{1}{N} Tr(H(\alpha)) \qquad \forall \alpha \in \mathfrak{G}$$

and extend it to the Wilson algebra $hol(M,G)$ by posing:

$$\varphi_H(p) := p_0 + \sum_{j_1} p_{j_1} \varphi_H(T_{\alpha_{j_1}}) + \sum_{j_1,j_2} p_{j_1,j_2} \varphi_H(T_{\alpha_{j_1}}) \varphi_H(T_{\alpha_{j_2}}) + \dots$$

To verify that this definition is well posed observe that any given Wilson polynomial p depends only on a finite number of elements $\alpha_1, \dots, \alpha_r$ in G, hence, as a consequence of the interpolation property, there exists a connection A (in general depending on p) such that

$$\varphi_H(T_{\alpha_k}) = \varphi_{H_A}(T_{\alpha_k}) = \frac{1}{N} Tr(H_A(\alpha_k)) = T_{\alpha_k}(A) \qquad k = 1, \dots, r.$$

Since the Wilson functions are well defined on equivalence classes of connections (or, equivalently, on unitary classes of holonomy maps), every functional φ_H is well defined.

The next step is to prove that φ_H is multiplicative and bounded.

Fix an arbitrary couple p_1, p_2 of Wilson polynomials and use again the interpolation property to choose a connection A such that $\varphi_H(\alpha_l) = \frac{1}{N} Tr(H_A(\alpha_l)) = T_{\alpha_l}(A)$ for every $\alpha_l \in \mathfrak{G}$ from which p_1 and p_2 depend.

From the multiplicative character of H_A one easily obtains $\varphi_H(p_1 p_2) = p_1(H_A) p_2(H_A) = \varphi_H(p_1)\varphi_U(p_2)$, thus φ_H is multiplicative.

Furthermore $|\varphi_H(p)| = |p(H_A)| \leq \|p\|$, hence φ_H is also bounded, alias continuous, so it can be extended in a unique way to a bounded multiplicative functional φ on the C^*-algebra obtained by the completion of $hol(M,G)$, but then φ belongs to $\sigma(Hol(M,G))$ and the theorem is proved. □

It is worth noting that $Hom(L_\star(M),G)/Ad_G$ is endowed with any particular topology, this one can be induced from that of $\sigma(Hol(M,G))$ only after their set-theoretical identification through the theorem just proved.

The results just proved can be resumed in the following important theorem.

Theorem 4.4.4 (Ashtekar-Lewandowski-Baumgärtel) *Whenever G is $U(N)$ or $SU(N)$, the spectrum of the holonomy C^*-algebra $Hol(M,G)$ can be algebraically characterized as the space of all homomorphism from the group of loops $L_\star(M)$ to G modulo conjugation:*

$$\sigma(Hol(M,G)) \simeq Hom(L_\star(M),G)/Ad_G\,.$$

Because of the fact that $\mathcal{A}/\mathcal{G}$ is densely embedded in $\sigma(Hol(M,G))$, in literature it has been chosen the symbol $\overline{\mathcal{A}/\mathcal{G}}$ to shortly denote the spectrum of the holonomy C^*-algebra, so that:

$$\overline{\mathcal{A}/\mathcal{G}} := \sigma(Hol(M,G)) \simeq Hom(L_\star(M),G)/Ad_G$$

the elements of $\overline{\mathcal{A}/\mathcal{G}}$ will be denoted by $\bar{A}$ and called **generalized connections**.

A generalized connection hence can be thought as a character of the holonomy C^*-algebra or as an algebraic homomorphisms from the group of loops to the gauge group modulo Ad_G-equivalence.

Since every connection on every PFB with base M and structure group G induces a holonomy map which happens to be a homomorphism from $L_\star(M)$ to G, the Ashtekar-Lewandowski-Baumgärtel theorem implies that **to every connection on every principal fiber bundle over M with structural group G corresponds a point in $\overline{\mathcal{A}/\mathcal{G}}$!**

Even though this space seems to be 'too big' to be interesting, it is 'small enough' to be endowed with a natural probability measure and a rich differential structure, these are two of the most important reason which makes $\overline{\mathcal{A}/\mathcal{G}}$ a natural candidate to the role of quantum configuration space of a gauge theory.

The algebraic characterization of $\overline{\mathcal{A}/\mathcal{G}}$ has this important corollary.

Corollary 4.4.1 *Whenever the gauge group G is $U(N)$ or $SU(N)$, $Hol(M,G)$ doesn't depend on the principal fiber bundle $P(M,G)$ but only on M and G.*

Proof. Since the spectrum $\overline{\mathcal{A}/\mathcal{G}}$ of $Hol(M,G)$ is characterized as the set $Hom(L_\star(M),G)/Ad_G$, it depends only on M and G, because this is the only dependence of an element of the last space.

The Gelfand isomorphism now reads:

$$\begin{array}{rccc} \hat{}: & Hol(M,G) & \longrightarrow & \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}}) \\ & f & \mapsto & \hat{f}, \end{array}$$

$\hat{f}(\bar{A}) := \bar{A}(f)$, thus even $Hol(M,G)$ doesn't depend on $P(M,G)$ but only on M and G, since it is isomorphic to $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$ which obviously depends only on M and G. □

The function f which appear in the previous proof is a Wilson function or a uniform limit of a sequence of Wilson functions. If f happens to be a Wilson functions T_α, for a certain loop $\alpha \in L_\star(M)$, then it is possible to write down the explicit action of its Gelfand transformed $\hat{T}_\alpha \in \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$, $\hat{T}_\alpha(\bar{A}) := \bar{A}(T_\alpha)$.

In fact it is sufficient to remember that, as shown in the proof of the theorem 4.4.3, the character $\bar{A} \in \sigma(Hol(M,G))$ is in bijection with a Ad_G-equivalence class $[H] \in Hom(L_\star(M),G)/Ad_G$ such that $\bar{A}(T_\alpha) = \frac{1}{N}Tr(H(\alpha))$, for every fixed representative $H \in [H]$, hence:

$$\hat{T}_\alpha(\bar{A}) = \frac{1}{N}Tr(H(\alpha))\,.$$

In the sequel, when there won't be risk of confusion, I shall omit the hat symbol and denote the functions of $Hol(M,G)$ and their Gelfand transformed − which are elements of $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$ − with the same symbol.

Chapter 5

The projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$ and its cylindrical measures

In this chapter I shell examine the projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$. This is a very useful (and elegant) identification of the spectrum of the holonomy C^*-algebra with a projective limit.

This result has clarified the structure of $\overline{\mathcal{A}/\mathcal{G}}$ and, most important, has enabled to construct a non-trivial measure on it, the so-called *uniform measure*, which is indispensable for the procedure of quantization of gauge theories that will be described in the following chapter.

The chapter is organized as follows: in the first section are collected the definitions and results about projective and inductive limits necessary to develop the projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$, which is reported in the second section. The third section is devoted to show how the uniform measure arises within the context of this characterization.

5.1 Projective and inductive limits

The basic definition about projective limits of topological spaces is the following (for a wider and more complete discussion on projective limits the interest reader is referred to [16]).

Def. 5.1.1 *A* **projective family** *of topological Hausdorff spaces is a triple* $\{\Omega_j, \pi_{ij}, J\}$ *where:*

- Ω_j *is a topological Hausdorff space for every* $j \in J$*.*

- *J is a* **directed** *set of indexes, i.e. it is endowed with a partial order relationship $\leq$ such that*

$$\forall i, j \in J \ \exists k \in J \text{ such that } i \leq k \text{ and } j \leq k.$$

- *if $i \leq j$ then the maps $\pi_{ij} : \Omega_j \to \Omega_i$ are continuous surjective projections such that:*

  1. *$\pi_{jj} = id_{\Omega_j} \ \forall j \in J$;*
  2. *if $i \leq j \leq k$ then $\pi_{ij} \circ \pi_{jk} = \pi_{ik}$ (**"consistency relation"**).*

An element $\{\omega_j\}_{j\in J}$ of the cartesian product $\prod_{j\in J} \Omega_j$ is called **wire** *if it satisfies the condition*

$$\pi_{ij}\omega_j = \omega_i \qquad \forall i < j$$

i.e. if every element of the ordered sequence is obtained from one of the previous via projection.

The **projective limit** *of $\{\Omega_j, \pi_{ij}, J\}$ is the subset of the cartesian product $\prod_{j\in J} \Omega_j$ given by all its wires, this space is indicated by*

$$\Omega \equiv \varprojlim_{j\in J} \Omega_j \,.$$

The maps

$$\begin{array}{rccl} \pi_j : & \Omega & \longrightarrow & \Omega_j \\ & \{\omega_i\}_{i\in J} & \mapsto & \pi_j(\{\omega_i\}_{i\in J}) := \omega_j \end{array}$$

are called the projections of Ω.

The projective limit Ω carries a natural topology, called **initial topology**, which is the smallest topology w.r.t. the projections π_j of Ω are continuous.

A base of this topology is given by the sets $\prod_{j\in J} U_j$, where $U_j \in \Omega_j$ is an open set such that $U_j = \Omega_j \ \forall j \in J$ but for a finite number of indexes.

In the initial topology *the projections are open maps* and *the projective limit is closed.*

It is easy to proof that if I is a **cofinal** subset of J, i.e. $\forall j \in J \ \exists i \in I$ such that $j \leq i$, then

$$\varprojlim_{j\in J} \Omega_j = \varprojlim_{i\in I} \Omega_i \,.$$

Furthermore, *if the spaces Ω_j are all compact then the projective limit Ω is a non-empty compact Hausdorff space.*

There is a very important class of functions associated to the projective limit of topological spaces, the class of the **cylindrical functions**.

Def. 5.1.2 *The space $Cyl(\Omega)$ of the cylindrical functions on the projective limit Ω of the family $\{\Omega_j, \pi_{ij}, J\}$ is the quotient of the disjoint union $\coprod_{j\in J}\mathcal{C}(\Omega_j)$ modulo the equivalence relation defined by: $f \in \mathcal{C}(\Omega_j)$, $g \in \mathcal{C}(\Omega_{j'})$, $f \sim g$ if there exists an index j'' such that $\pi_{jj''}(f) = \pi_{j'j''}(g)$.*

Note that, in particular, the cylindrical functions are continuous, by converse it can be easily proved that *a continuous function f on Ω is cylindrical if and only if there exists a function $f_j \in \mathcal{C}(\Omega_j)$ such that $f = f_j \circ \pi_j$*, if this is the case then f is said to be cylindrical w.r.t. the index j and one writes $f \in Cyl_j(\Omega)$. Obviously

$$Cyl(\Omega) = \coprod_{j\in J} Cyl_j(\Omega)\,.$$

The map

$$\begin{array}{rccl} i: & Cyl(\Omega) & \longrightarrow & \mathcal{C}(\Omega) \\ & f_j & \mapsto & i(f_j) := f_j \circ \pi_j \end{array}$$

is an injective homomorphism which embeds $Cyl(\Omega)$ in $\mathcal{C}(\Omega)$.

The final result I want to cite about projective limits is the celebrated A.Weil's theorem (see [41]).

Theorem 5.1.1 *Every compact group is the projective limit of compact Lie groups.*

The dual construction of the projective limit is the inductive limit. For the later purposes it is worth introducing the definition of inductive limit directly on C^*-algebras, the same definition extends to general linear spaces and algebras. Here the reference is [29].

Def. 5.1.3 *An* **inductive family** *of C^*-algebras is a triple $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ where $\mathfrak{A}_\alpha$ are C^*-algebras and A is a directed set of indexes such that, for every $\alpha \le \beta$, there exist continuous injective inclusions $i_{\beta\alpha} : \mathfrak{A}_\alpha \to \mathfrak{A}_\beta$ satisfying:*

1. $i_{\alpha\alpha} = id_{\mathfrak{A}_\alpha}$;
2. $i_{\gamma\beta} \circ i_{\beta\alpha} = i_{\gamma\alpha}$, *whenever* $\alpha \le \beta \le \gamma$.

The **inductive limit** *of $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ is, set-theoretically, the quotient of the disjoint union $\coprod_{\alpha\in A}\mathfrak{A}_\alpha$ modulo the following equivalence relation: $a \in \mathfrak{A}_\alpha$, $b \in \mathfrak{A}_\beta$, $a \sim b$ if there exists $\gamma \ge \alpha, \beta$ such that $i_{\gamma\alpha}(a) = i_{\gamma\beta}(b)$.*

The symbol used to represent the inductive limit is

$$\mathfrak{A} \equiv \varinjlim_{\alpha\in A}\mathfrak{A}_\alpha\,.$$

The canonical inclusion of $\mathfrak{A}_\alpha$, α fixed in A, in the disjoint union defines, by quotient, the inclusion map in the inductive limit $\mathfrak{A}$, $i_\alpha : \mathfrak{A}_\alpha \hookrightarrow \mathfrak{A}$, which satisfies $i_\beta \circ i_{\beta\alpha} = i_\alpha$ for every $\alpha \leq \beta$.

To endow $\mathfrak{A}$ with an algebraic structure it is necessary to use the following lemma.

Lemma 5.1.1 *Let $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ be an inductive family of C^*-algebras with inductive limit $\mathfrak{A}$. Then, fixed n elements $\{a_1, \ldots, a_n\} \subset \mathfrak{A}$, there exist an index β and n elements $\{b_1, \ldots, b_n\} \subset \mathfrak{A}_\beta$ such that*

$$a_i = i_\beta(b_i) \qquad i = 1, \ldots, n.$$

Thanks to the previous lemma one can define the $*$-algebraic structure of $\mathfrak{A}$ using that of the $*$-algebras appearing in the family:

$$\begin{cases} \lambda a := i_\beta(\lambda b) \\ a_1 + a_2 := i_\beta(b_1 + b_2) \\ a_1 a_2 := i_\beta(b_1 b_2) \\ a^* := i_\beta(b^*) \end{cases}$$

where $\lambda \in \mathbb{C}$, $a, a_1, a_2 \in \mathfrak{A}$ and $b, b_1, b_2 \in \mathfrak{A}_\beta$ satisfy $i_\beta(b) = a$, $i_\beta(b_1) = a_1$ and $i_\beta(b_2) = a_2$.

By endowing $\mathfrak{A}$ of the finest locally convex topology which makes the homomorphisms i_α continuous, called **final topology**, $\mathfrak{A}$ becomes a topological $*$-algebra.

It is worth noting that **an inductive family of C^*-algebras always induces a projective family**, in fact if $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ is such a family then a projective family is obtained by associating to every $\mathfrak{A}_\alpha$ its spectrum $\sigma(\mathfrak{A}_\alpha)$ and to every inclusion $i_{\beta\alpha}$, $\alpha \leq \beta$, the restriction of its transposed map to the spectrum of $\mathfrak{A}_\beta$, $\pi_{\alpha\beta} \equiv {}^t i_{\beta\alpha}|_{\sigma(\mathfrak{A}_\beta)}$, where:

$$\begin{array}{rccc} {}^t i_{\beta\alpha} : & {\mathfrak{A}_\beta}^* & \longrightarrow & {\mathfrak{A}_\alpha}^* \\ & \varphi & \mapsto & {}^t i_{\beta\alpha}(\varphi), \end{array}$$

is defined in the usual way, i.e. $({}^t i_{\beta\alpha}(\varphi))(a) := \varphi(i_{\beta\alpha}(a))$, $a \in \mathfrak{A}_\alpha$.

It is easy to verify that the family $\{\sigma(\mathfrak{A}_\alpha), \pi_{\alpha\beta}, A\}$ is a well defined projective family.

If the $\mathfrak{A}_\alpha$ are also unital and Abelian then the spectra $\sigma(\mathfrak{A}_\alpha)$ are compact Hausdorff spaces, hence the projective limit $\varprojlim_{\alpha \in A} \mathfrak{A}_\alpha$ is a non-void compact Hausdorff space.

The most remarkable fact about this family, which will be used in the next section to obtain the projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$, is expressed by the following theorem (Th. 3.43 of [29]).

Theorem 5.1.2 *Let $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ be an inductive family of Abelian C^*-algebras with unit. Then its inductive limit $\mathfrak{A}$ is an Abelian topological algebra with unit (in the final topology) whose spectrum $\sigma(\mathfrak{A})$ is a compact Hausdorff space homeomorphic to the projective limit of $\{\sigma(\mathfrak{A}_\alpha), \pi_{\beta\alpha}, A\}$:*

$$\mathfrak{A} = \varinjlim_{\alpha\in A} \mathfrak{A}_\alpha \quad \Rightarrow \quad \sigma(\mathfrak{A}) \simeq \varprojlim_{\alpha\in A} \sigma(\mathfrak{A}_\alpha)\,.$$

5.2 The projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$

In this section it will be shown how $\overline{\mathcal{A}/\mathcal{G}}$ can be identified with a projective limit.

First of all fix the directed set of indexes to be the set of all graphs Γ in M ordered w.r.t. the natural inclusion and denote it by L. This set is directed because if Γ and Γ' belong to L then also $\Gamma \cup \Gamma'$ belongs to L and $\Gamma \le \Gamma \cup \Gamma'$, $\Gamma' \le \Gamma \cup \Gamma'$.

Now the idea is to use this directed set to construct an inductive family of C^*-algebras whose inductive limit is dense in the holonomy C^*-algebra, then, by using theorem 5.1.2, the desired result will be reached.

To every graph Γ associate the unital Abelian C^*-algebra $A(\Gamma)$ generated by the Wilson functions T_α such that $\alpha^* \subset \Gamma$ for, at least, one representative loop in $[\alpha]_{el} \in L_\star(M)$.

It is obvious that if $f \in A(\Gamma)$ then $f \in A(\Gamma')$ for every $\Gamma' \ge \Gamma$ so that the inclusions $i_{\Gamma'\Gamma}$ are naturally defined by:

$$\begin{array}{rccl} i_{\Gamma'\Gamma}: & A(\Gamma) & \hookrightarrow & A(\Gamma') \\ & f & \mapsto & i_{\Gamma'\Gamma}(f) := f\,. \end{array}$$

This inclusions satisfy the consistency relations: $i_{\Gamma''\Gamma'} \circ i_{\Gamma'\Gamma} = i_{\Gamma''\Gamma}$ for every $\Gamma'' \ge \Gamma' \ge \Gamma$.

Moreover the inclusion $i_\Gamma : A(\Gamma) \hookrightarrow Hol(M,G)$, $i_\Gamma(f) := f$, satisfies $i_\Gamma = i_{\Gamma'} \circ i_{\Gamma'\Gamma}$.

Hence $\{A(\Gamma), i_{\Gamma'\Gamma}, L\}$ is an inductive family of unital Abelian C^*-algebras whose inductive limit is continuously included in $Hol(M,G)$.

By comparing the definition of inductive limit of the $A(\Gamma) \subset \mathcal{C}(\mathcal{A}/\mathcal{G})$ and the definition of the algebra of the cylindrical functions on $\mathcal{A}/\mathcal{G}$ one immediately recognizes that the two algebras agree:

$$\varinjlim_{\Gamma\in L} A(\Gamma) = Cyl(\mathcal{A}/\mathcal{G})\,.$$

Observe now that the polynomial algebra $\mathcal{W}$ generated by the Wilson functions is contained in $Cyl(\mathcal{A}/\mathcal{G})$ hence:

$$\overline{Cyl(\mathcal{A}/\mathcal{G})} = Hol(M,G)\,.$$

If $\sigma(\Gamma)$ denotes the (compact, Hausdorff) spectrum of $A(\Gamma)$, then the theorem 5.1.2 implies that

$$\varprojlim_{\Gamma \in L} \sigma(\Gamma) = \sigma(Cyl(\mathcal{A}/\mathcal{G}))$$

where the projective limit is referred to the family $\{\sigma(\Gamma), \pi_{\Gamma\Gamma'}, L\}$, with $\pi_{\Gamma\Gamma'} := {}^t i_{\Gamma'\Gamma}|_{\sigma(\Gamma')}$.

The last step before the theorem of characterization of $\overline{\mathcal{A}/\mathcal{G}}$ as a projective limit is given by the next theorem.

Theorem 5.2.1 *The following assertions hold.*

1. *Let φ be a continuous linear functional on $Hol(M,G)$, then its restriction $\varphi_\Gamma := \varphi|_{A(\Gamma)}$ is a continuous linear functional on $A(\Gamma)$ and the family $\{\varphi_\Gamma\}(\Gamma \in L)$ satisfies the following conditions:*

   $$i)\ \varphi_{\Gamma'}|_{A(\Gamma)} = \varphi_\Gamma \qquad \forall \Gamma' \geq \Gamma;$$

   ii) the collection $\{\|\varphi_\Gamma\|\}(\Gamma \in L)$ admits a finite maximum.

   The property i) is called **consistency**, *the property ii) instead expresses the* **uniform boundness** *of the family $\{\varphi_\Gamma\}(\Gamma \in L)$;*

2. *If φ is a state of $Hol(M,G)$, then φ_Γ is a state of $A(\Gamma)$, for every $\Gamma \in L$;*

3. *If φ is a character of $Hol(M,G)$, then φ_Γ is a character of $A(\Gamma)$, for every $\Gamma \in L$;*

4. *By converse, a family $\{\varphi_\Gamma\}(\Gamma \in L)$ of continuous linear functionals (resp. states, characters) of the C^*-algebras $A(\Gamma)$ satisfying the conditions i) and ii) of 1. defines a continuous linear functional φ (resp. a state, a character) on $Hol(M,G)$ whose restriction to $A(\Gamma)$ is precisely φ_Γ, for every $\Gamma \in L$.*

Proof. 1. The consistency is obvious, the uniform boundness is shown simply by the observation that for every $\Gamma \in L$ one has:

$$\|\varphi_\Gamma\| = \sup_{f \in A(\Gamma), \|f\|=1} |\varphi(f)| \leq \|\varphi\| < +\infty.$$

2. The norm of the state φ is $\|\varphi\| = \varphi(\mathbf{1}) = 1$, thus every restriction φ_Γ is a positive linear functional on $A(\Gamma)$ and also $\varphi_\Gamma(\mathbf{1}) = \varphi(\mathbf{1}) = 1$, i.e. φ_Γ is a state of $A(\Gamma)$.

3. Obvious, the characters are the multiplicative states.

4. Consider first the polynomial algebra $\mathcal{W}$ generated by the Wilson functions and define on it the functional:

$$\begin{array}{rccc} \varphi_0 : & \mathcal{W} & \longrightarrow & \mathbb{C} \\ & p & \mapsto & \varphi_0(p) := \varphi_\Gamma(p) \end{array}$$

where Γ is any graph containing all the images of the loops which label the Wilson functions generating the polynomial

$$p = p_0 + \sum_{j_1} p_{j_1} T_{\alpha_{j_1}} + \sum_{j_2} p_{j_1,j_2} T_{\alpha_{j_1}} T_{\alpha_{j_2}} + \dots .$$

The functional φ_0 is *well defined*, in fact, thanks to the consistency of the family $\{\varphi_\Gamma\}(\Gamma \in L)$, if Γ' is another graph containing the images of the loops above, then $\varphi_\Gamma(p) = \varphi_{\Gamma\cap\Gamma'}(p) = \varphi_{\Gamma'}(p)$.

φ_0 is *linear*, in fact if $p, q \in \mathcal{W}$ then it certainly exists a graph Γ such that $p + q \in A(\Gamma)$, hence, thanks to the linearity of φ_Γ on $A(\Gamma)$:

$$\begin{cases} \varphi_0(p+q) = \varphi_\Gamma(p+q) = \varphi_\Gamma(p) + \varphi_\Gamma(q) = \varphi_0(p) + \varphi_0(q); \\ \varphi_0(\lambda p) = \varphi_\Gamma(\lambda p) = \lambda\varphi_\Gamma(p) = \lambda\varphi_0(p), \ \lambda \in \mathbb{C}. \end{cases}$$

φ_0 is *bounded*, this follows from the boundness of φ_Γ:

$$|\varphi_0(p)| = |\varphi_\Gamma(p)| \leq \|\varphi_\Gamma\|\|p\|$$

for every $p \in \mathcal{W}$.

Being $\mathcal{W}$ dense in $Hol(M, G)$, thanks to the theorem of bounded extension of bounded linear functionals, φ_0 can be extended to a unique bounded linear functional φ on $Hol(M, G)$ whose restriction to every $A(\Gamma)$ is φ_Γ.

To prove that φ is a state when the functionals φ_Γ are states it is previously necessary to observe that if the functionals φ_Γ are positive for every Γ then also φ is positive. In fact, since $\mathcal{W} \subset Cyl(\mathcal{A}/\mathcal{G})$, then $\varphi(p) \geq 0$ for every $p \geq 0$, $p \in \mathcal{W}$, thanks to the positivity of the states φ_Γ. Moreover every $f \in Hol(M, G)$, $f \geq 0$, is the uniform limit of a sequence of positive polynomials, this is easy to verify by taking the square root $\sqrt{f}$ of f (it certainly exists because f is positive!) and by writing it as the uniform limit of Wilson polynomials: $\sqrt{f} = \lim_n p_n$. By definition of square root, $f = \lim_n {p_n}^* p_n$ and ${p_n}^* p_n \geq 0$ for every $n \in \mathbb{N}$, thus $\varphi(f) = \lim_n \varphi({p_n}^* p_n) \geq 0$ by virtue of the theorem of persistence of the signum.

Now, if $\{\varphi_\Gamma\}(\Gamma \in L)$ is a family of states of the C^*-algebras $A(\Gamma)$, then φ is a state of $Hol(M, G)$, in fact the states are positive and so φ is positive (for what just shown), hence its norm is the value assumed in the unit element of the algebra: $\|\varphi\| = \varphi(\mathbf{1}) = \varphi_\Gamma(\mathbf{1}) = 1$, for all $\Gamma \in L$.

Finally if the functionals φ_Γ are characters of the algebras $A(\Gamma)$, then φ_0 is multiplicative on $\mathcal{W}$ and so, written the functions $f, g \in Hol(M,G)$ as $f = \lim_n p_n$, $g = \lim_n q_n$, with $\{p_n\}, \{q_n\}(n \in \mathbb{N}) \subset \mathcal{W}$, one has:

$$\varphi(fg) = \lim_n \varphi_0(p_n q_n) = \lim_n \varphi_0(p_n)\varphi_0(q_n) = \lim_n \varphi(p_n)\varphi(q_n)$$

because φ and φ_0 act in the same way on $\mathcal{W}$.

Thanks to the continuity of φ it follows that:

$$\varphi(fg) = \varphi(\lim_n p_n)\,\varphi(\lim_n q_n) = \varphi(f)\varphi(g)$$

showing that φ is a character. □

Theorem 5.2.2 (Projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$) *The spectrum of $Hol(M,G)$ is homeomorphic to the projective limit of the family $\{\sigma(\Gamma), \pi_{\Gamma\Gamma'}, L\}$ w.r.t. the initial topology:*

$$\overline{\mathcal{A}/\mathcal{G}} \simeq \varprojlim_{\Gamma \in L} \sigma(\Gamma)\,.$$

Proof. First of all observe that there is a set-theoretical bijection between $\overline{\mathcal{A}/\mathcal{G}}$ and $\varprojlim_{\Gamma\in L}\sigma(\Gamma)$ because $Cyl(\mathcal{A}/\mathcal{G})$ is dense in $Hol(M,G)$, hence the characters of $Cyl(\mathcal{A}/\mathcal{G})$ can be univocally extended to characters of $Hol(M,G)$ and, by converse, the characters of $Hol(M,G)$ reduces to characters of $Cyl(\mathcal{A}/\mathcal{G})$ simply by restriction, thus these C^*-algebras have the same spectrum.

But then, from 3.and 4.of the previous theorem and from theorem 5.1.2, it follows that $\varprojlim_{\Gamma\in L}\sigma(\Gamma) = \sigma(Cyl(\mathcal{A}/\mathcal{G})) = \sigma(Hol(M,G))$, set-theoretically.

Since the Wilson functions generate $Hol(M,G)$, the topology on its spectrum $\overline{\mathcal{A}/\mathcal{G}}$ is the initial topology defined as the smallest topology in which the (Gelfand transformed of the) Wilson functions are continuous.

For the same reason the spectra $\sigma(\Gamma)$ have the initial topology defined as the smallest topology in which the Wilson functions T_α, with $\alpha^* \subset \Gamma$, are continuous.

By remembering that a projective limit of compact Hausdorff spaces inherits the initial topology the theorem follows. □

Pictorially, the duality between the inductive family of C^*-algebras $A(\Gamma)$ and the projective family of their spectra $\sigma(\Gamma)$ can be represented as follows:

$$\ldots \subseteq A(\Gamma) \subseteq \ldots \subseteq A(\Gamma') \subseteq \ldots \longrightarrow \varinjlim_{\Gamma\in L} A(\Gamma) \equiv Cyl(\mathcal{A}/\mathcal{G})$$

$$\ldots \supseteq \sigma(\Gamma) \supseteq \ldots \supseteq \sigma(\Gamma') \supseteq \ldots \longleftarrow \varprojlim_{\Gamma\in L}\sigma(\Gamma) \equiv \overline{\mathcal{A}/\mathcal{G}}\,.$$

5.2.1 The characterization of the spectra $\sigma(\Gamma)$

The spectra $\sigma(\Gamma)$ can be explicitly characterized in a useful way by using a few topological results.

Let Γ be a *connected graph* and $\star$ a fixed point of Γ, then its **fundamental group**, or **first homotopy group**, $\pi_1(\Gamma, \star)$, is the group of the homotopy classes of loops based on $\star$ with image contained in Γ.

Since Γ is assumed to be connected, the choice of the (fixed) base-point $\star$ is irrelevant in the definition of the fundamental group, thus I shell denote it simply by $\pi_1(\Gamma)$.

It is well known (see for instance ch.14 of [17]) that $\pi_1(\Gamma)$ *is a free group with* n_Γ *generators*, where n_Γ is the **connectivity** of Γ, i.e. the integer:

$$n_\Gamma = E_\Gamma - V_\Gamma + 1$$

being E_Γ the number of edges of Γ and V_Γ the number of its vertexes; n_Γ *is a topological invariant* of the graph Γ which represents the highest number of edges that can be deleted from the graph without it fails to be connected.

Denote with $L_\star(\Gamma)$ the subgroup of $L_\star(M)$ given by the loops containing at least a representative α with $\alpha^* \subset \Gamma$.

Theorem 5.2.3 *For every graph* Γ *the following assertions hold.*

1. *The group* $L_\star(\Gamma)$ *is isomorphic to* $\pi_1(\Gamma)$*;*
2. *The generators* $\beta_1, \ldots, \beta_{n_\Gamma}$ *of* $\pi_1(\Gamma)$ *form an independent family of loops in* $L_\star(M)$*;*
3. *For every fixed graph* Γ *the following spaces are homeomorphic:*

$$\sigma(\Gamma) \simeq Hom(L_\star(\Gamma), G)/Ad_G \simeq G^{n_\Gamma}/Ad_G\,.$$

The proof of this theorem can be found in [1], here the only important thing is that the explicit form of the isomorphism in 3. is given by the following map:

$$\begin{array}{rccl} \phi_{\vec{\beta}}: & G^{n_\Gamma}/Ad_G & \longrightarrow & Hom(L_\star(\Gamma), G)/Ad_G \\ & [g_1, \ldots, g_{n_\Gamma}] & \mapsto & \phi_{\vec{\beta}}([g_1, \ldots, g_{n_\Gamma}]) := [H_{\vec{g}}] \end{array}$$

where $\vec{\beta} \equiv \{\beta_1, \ldots, \beta_{n_\Gamma}\}$ is a fixed family of generators of $\pi_1(\Gamma)$ and $H_{\vec{g}}(\beta_i) := g_i$, $i = 1, \ldots, n_\Gamma$, $\vec{g} = (g_1, \ldots, g_{n_\Gamma})$. It is easy to see that $\phi_{\vec{\beta}}$ is invertible and that its inverse is the evaluation map in the generators of $\pi_1(\Gamma)$, i.e. $\phi_{\vec{\beta}}^{-1} = ev(\beta_1, \ldots, \beta_{n_\Gamma})$.

Since the evaluation map is certainly continuous, the only thing that remains to do is to prove that $\phi_{\vec{\beta}}$ is continuous.

Remember that, if $[\tilde{H}]$ is a fixed element of $Hom(L_\star(\Gamma), G)/Ad_G$, then a base of open neighborhoods of $[\tilde{H}]$ is given by:

$$U := \{[H] \ : \ \frac{1}{N}|Tr(H(\alpha_i)) - Tr(\tilde{H}(\alpha_i))| < \varepsilon, \ i = 1, \dots, k\}$$

for a given finite set of loops α_i, $i = 1, \dots, k$.

The proof of the continuity of $\phi_{\vec{\beta}}$ is equivalent to the proof that $\phi_{\vec{\beta}}^{-1}(U)$ is open.

This fact certainly holds when $\alpha_i = \beta_i$, $i = 1, \dots, k$, in fact in this situation $Tr(H_{\vec{g}}(\alpha_i)) = Tr(H_{\vec{g}}(\beta_i)) = Tr(g_i)$, and, being Tr a continuous function, the set of the $(g_1, \dots, g_n)$ such that $|Tr(g_i) - \lambda_i| < \varepsilon$ is an open set in G^{n_Γ} for every $\lambda_i \in \mathbb{C}$.

Now decompose every α_i as

$$\alpha_i = {\beta_1}^{m^i_{1,1}} \cdots {\beta_{n_\Gamma}}^{m^i_{n_\Gamma,1}} {\beta_1}^{m^i_{1,2}} \cdots {\beta_{n_\Gamma}}^{n^i_{n_\Gamma,2}} \cdots$$

then

$$Tr(H_{\vec{g}}(\alpha_i)) = Tr({g_1}^{m^i_{1,1}} \cdots {g_{n_\Gamma}}^{m^i_{n_\Gamma,1}} {g_1}^{m^i_{1,2}} \cdots {g_{n_\Gamma}}^{n^i_{n(\Gamma,2)}} \cdots)$$

since $H_{\vec{g}}$ is a homomorphism and so factorizes as the loops.

The continuity of the trace implies again that the set of the $(g_1, \dots, g_{n_\Gamma})$ such that $|Tr({g_1}^{m^i_{1,1}} \cdots {g_{n_\Gamma}}^{m^i_{n_\Gamma,1}} {g_1}^{m^i_{1,2}} \cdots {g_{n_\Gamma}}^{n^i_{n(\Gamma,2)}} \cdots) - \lambda_i| < \varepsilon$ is open in G^{n_Γ} for every $\lambda_i \in \mathbb{C}$.

Hence, taking in particular $\lambda_i = Tr(\tilde{H}(\alpha_i))$, for every $i = 1, \dots, k$, one has the thesis.

Finally, the homeomorphism between $Hom(L_\star(\Gamma), G)/Ad_G$ and $\sigma(\Gamma)$ is just a consequence of the results discussed in 4.4 applied to the C^*-algebras $A(\Gamma)$ instead of the holonomy C^*-algebra $Hol(M, G)$. □

As an immediate corollary of the previous theorem one obtains this explicit and useful projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$:

$$\overline{\mathcal{A}/\mathcal{G}} \simeq \varprojlim_{\Gamma \in L} G^{n_\Gamma}/Ad_G \,.$$

5.3 Cylindrical measures on $\overline{\mathcal{A}/\mathcal{G}}$

The references for this sections are [28] and [44].

First of all remember that a probability space is a triple (Ω, Σ, p), where Ω is a non-empty measurable space, Σ is a σ-algebra on Ω and p is a probability measure on Ω: $p(\Omega) = 1$.

If T is any index set, one can consider the family $\{(\Omega_t, \Sigma_t, p_t\}(t \in T)$ of measurable spaces and define the *product σ-algebra* $\prod_{t\in T} \Sigma_t$ on the cartesian product $\prod_{t\in T} \Omega_t$ to be the smallest σ-algebra containing the cylindrical subsets $\prod_{t\in T} M_t \subset \prod_{t\in T} \Omega_t$, where $\prod_{t\in T} M_t$ is said to be cylindrical if

1. $M_t \in \Sigma_t$, $\forall t \in T$;

2. $M_t = \Omega_t$ but for a finite subset of indexes in T.

This σ-algebra is the smallest which makes the projections $p_r : \prod_{t\in T} \Omega_t \to \Omega_r$ measurable maps, for every fixed $r \in T$; one says shortly that this is the σ-algebra generated by the projections.

The probability measure p defined on the cylindrical subsets by:

$$p(\prod_{t\in T} M_t) := \prod_{t\in T} p_t(M_t)$$

is called, for obvious reasons, **cylindrical measure** and it always extends to a measure defined on the entire product σ-algebra.

Now take, in particular, a directed set J and suppose that the family of probability spaces $\{\Omega_j\}(j \in J)$ has measurable projections $\pi_{jj'}$, defined for every $j \le j'$ and satisfying the axioms of a projective family, then the triple $\{\Omega_j, \pi_{jj'}, J\}$ is said to be a **projective family of probability spaces**.

Suppose now to have a measure μ on the projective limit Ω of this family, then the push-forward of μ via the canonical projection $\pi_j : \Omega \to \Omega_j$, i.e. $\mu_j := \pi_{j*}\mu \equiv \mu \circ \pi_j$, is a measure on Ω_j, for every $j \in J$.

Furthermore the family of measures $\{\mu_j\}(j \in J)$ satisfies the **consistency condition**

$$\mu_j = (\pi_{jj'})_* \mu_{j'} = \mu_{j'}|_{\Omega_j} \circ \pi_{jj'}$$

which guaranties that there is no ambiguity when a portion of Ω_j is measured directly by μ_j or by the restriction of $\mu_{j'}$ to Ω_j.

A family of measures $\{\mu_j\}(j \in J)$ satisfying the consistency condition is said to be a **promeasure**.

A classical problem of measure theory is to study when it is possible to construct a measure μ on Ω starting by a promeasure, i.e. when it is possible to obtain a representation theorem for measures on projective limits, since the inverse process is always possible, as just discussed.

When the index set J is numerable this representation theorem is available every time the spaces Ω_j are σ-compact metric spaces and the promeasure is Borel-like.

However the request of numerability of the index set J is quite restrictive, luckily when the probability spaces of the projective family are compact the extension of a promeasure to a measure on the projective limit is always possible.

Before formalizing this assertion in a theorem it is worth remembering once again that, when the spaces Ω_j of a projective family are compact Hausdorff spaces, the projective limit Ω is a non-empty compact Hausdorff space itself; furthermore, in this situation, the algebra of the cylindrical functions on Ω satisfies the hypothesis of the Stone-Weierstrass theorem and so it is dense in the algebra of the continuous complex-valued functions on Ω:

$$\overline{Cyl(\Omega)} = \mathcal{C}(\Omega) \qquad \text{(if } \Omega \text{ is compact).}$$

To simplify the notation in the sequel a regular Borel probability measure will be simply called "probability measure".

Theorem 5.3.1 *Let $\{\Omega_j, \pi_{jj'}, J\}$ be a projective family of compact Hausdorff spaces with projective limit Ω.*

Then there is a bijective correspondence between probability measures on Ω and probability promeasures $\{\mu_j\}(j \in J)$.

All such measures are cylindrical.

Proof. It has to be proved that a probability promeasure $\{\mu_j\}(j \in J)$ univocally defines a probability measures on Ω.

Define the linear functional

$$\begin{array}{rccl} F: & \coprod_{j\in J} \mathcal{C}(\Omega_j) & \longrightarrow & \mathbb{C} \\ & f_j & \mapsto & F(f_j) := \int_{\Omega_j} f_j \, d\mu_j \end{array}$$

$\forall f_j \in \mathcal{C}(\Omega_j)$.

Thanks to the consistency condition of the measures appearing in the promeasure, F factorizes to $Cyl(\Omega)$ in a natural fashion.

Being bounded, F admits a unique extension to a bounded linear functional $\bar{F}$ on the closure of $Cyl(\Omega)$, i.e. on $\mathcal{C}(\Omega)$.

Thanks to the Riesz-Markov theorem it exists a unique probability measure μ on Ω which represents the functional $\bar{F}$ in the usual way, i.e. $\bar{F}(f) := \int_\Omega f \, d\mu$, $\forall f \in \mathcal{C}(\Omega)$.

The measure μ is obviously cylindrical. □

This result can be specialized to the projective family of the compact Hausdorff spaces $\{\sigma(\Gamma)\}(\Gamma \in L)$, which gives rise to the the compact Hausdorff space $\overline{\mathcal{A}/\mathcal{G}}$, to obtain the following important result.

Corollary 5.3.1 *There is a bijection between the probability measures on* $\overline{\mathcal{A}/\mathcal{G}}$ *and the probability promeasures* $\{\mu_\Gamma\}(\Gamma \in L)$ *on the spectra* $\sigma(\Gamma)$.

Thanks to the characterization $\sigma(\Gamma) \simeq G^{n_\Gamma}/Ad_G$ an explicit (and natural) promeasure which gives rise to a probability measure on $\overline{\mathcal{A}/\mathcal{G}}$ is given by the family of the normalized Haar measures dg^{n_Γ} on the groups G^{n_Γ}, which are Ad_G-invariant (thanks to the assumption of compactness for G) and thus projects unaffected to the quotient G^{n_Γ}/Ad_G.

The probability measure obtained from the promeasure $\{dg^{n_\Gamma}\}(\Gamma \in L)$ is called the **uniform measure** on $\overline{\mathcal{A}/\mathcal{G}}$ and denoted by μ_0.

If a function $f \in Hol(M,G) \simeq \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$ is cylindrical w.r.t. the index-graph Γ, i.e. it exists $f_\Gamma \in \mathcal{C}(G^{n_\Gamma}/Ad_G)$ such that $f = f_\Gamma \circ \pi_\Gamma$, then its explicit integral w.r.t. the uniform measure is given by:

$$\int_{\overline{\mathcal{A}/\mathcal{G}}} f(\bar{A})\, d\mu_0(\bar{A}) = \int_{G^{n_\Gamma}} f_\Gamma(g_1, \ldots, g_{n_\Gamma})\, dg^{n_\Gamma}(g_1, \ldots, g_{n_\Gamma})\,.$$

Thanks to the density of $Cyl(\overline{\mathcal{A}/\mathcal{G}})$ in $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$, the formula above extends (by uniform limit) to all the functions of $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$.

J.Baez has found in [10] a way to construct measures on $\overline{\mathcal{A}/\mathcal{G}}$ starting from random variables, his work contemplates the uniform measures as a particular case.

5.3.1 Diffeomorphism invariant measures on $\overline{\mathcal{A}/\mathcal{G}}$

Now I put the attention on the invariance of the measures on $\overline{\mathcal{A}/\mathcal{G}}$ under diffeomorphisms.

Let $\text{Diff}_0(M)$ be the group of the *analytical diffeomorphisms of M homotopic to the identity map.*

The action of $\text{Diff}_0(M)$ on $Hol(M,G)$ is defined in the following way:

$$\begin{array}{ccc} Hol(M,G) \times \text{Diff}_0(M) & \longrightarrow & Hol(M,G) \\ (T_\alpha, \Psi) & \mapsto & \Psi_* T_\alpha, \end{array}$$

$(\Psi_* T_\alpha)([A]) := T_{\Psi \circ \alpha}([A])$, for every $[A] \in \mathcal{A}/\mathcal{G}$.

This action is well defined because the Wilson functions generate $Hol(M,G)$.

Observe that $\Psi\circ\alpha$ is precisely the loop α deformed by the diffeomorphism Ψ.

It is useful to reformulate this action using the structure of principal fiber bundle.

In the section 1.2 we have seen that there is an injective homomorphism

$$\begin{array}{rccc} \flat: & \mathrm{Aut}(P) & \longrightarrow & \mathrm{Diff}(M) \\ & \Phi & \mapsto & \flat(\Psi)=\Psi \end{array}$$

whose range contains $\mathrm{Diff}_0(M)$, thus the action of $\mathrm{Diff}_0(M)$ on the Wilson functions can be reformulated as follows

$$\Psi_* T_\alpha([A]) = T_{\Psi\circ\alpha}([A]) = T_\alpha(\Phi^*([A]))$$

where Φ^* is the pull-back of the automorphism $\Phi\in\mathrm{Aut}(P)$ such that $\flat(\Phi)=\Psi$.

This action naturally extend to the Wilson polynomials p by the position $\Psi_* p([A]) := p(\Psi^*([A]))$.

Observe that the deformation of the loop α doesn't affect the norm of the Wilson functions, since the holonomy of the loop α is invariant under the diffeomorphisms of $\mathrm{Diff}_0(M)$. This property also extends to the Wilson polynomials: $\|\Psi_* p\| = \|p\|$.

Since the Wilson polynomials are dense in $Hol(M,G)$, the action of $\mathrm{Diff}_0(M)$ can be extended to an isometric action on the entire $Hol(M,G)$, i.e., using the same symbol

$$\begin{array}{rccc} \Psi_*: & Hol(M,G) & \longrightarrow & Hol(M,G) \\ & f & \mapsto & \Psi_* f := \lim_n \Psi_* p_n \end{array}$$

where $\{p_n\}(n\in\mathbb{N})\subset\mathcal{W}$, $f=\lim_n p_n$.

This fact gives the possibility to define a representation of $\mathrm{Diff}_0(M)$ in isometries of $Hol(M,G)$:

$$\begin{array}{rccc} \rho: & \mathrm{Diff}_0(M) & \longrightarrow & \mathrm{Aut}(Hol(M,G)) \\ & \Psi & \mapsto & \rho(\Psi) := \Psi_*. \end{array}$$

Def. 5.3.1 *A positive linear functional $\varphi\in Hol(M,G)^*$ is said to be* **invariant under diffeomorphisms** *if*

$$\varphi(\Psi_* f) = \varphi(f) \qquad \forall f\in Hol(M,G),\ \forall\Psi\in \mathit{Diff}_0(M).$$

A probability measure on $\overline{\mathcal{A}/\mathcal{G}}$ is called invariant under diffeomorphisms if its corresponding state φ_μ possesses this invariance.

If $\Psi_* \mu$ denotes the measure corresponding to the functional $\varphi_\mu \circ \Psi_*$ then the invariance of the measure μ is written symbolically as: $\Psi_* \mu = \mu$.

Since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, fixed a positive regular Borel measure μ on it, one has that $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu) \subset L^1(\overline{\mathcal{A}/\mathcal{G}}, \mu)$, hence the Radon-Nykodim theorem implies that, for every $f \in L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$, there exists one and only one measure μ_f which is absolutely continuous w.r.t. μ and such that f can be written as the Radon-Nykodim derivative: $f = \frac{d\mu_f}{d\mu}$.

This enables to construct a unitary representation of $\mathrm{Diff}_0(M)$ supported by $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$ and labelled by the positive regular Borel measure μ, this representation is defined by:

$$\begin{array}{rccl} U: & \mathrm{Diff}_0(M) & \longrightarrow & \mathcal{U}(L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)) \\ & \Psi & \mapsto & U_\Psi \end{array}$$

where

$$\begin{array}{rccl} U_\Psi: & L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu) & \longrightarrow & L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu) \\ & f & \mapsto & U_\Psi(f) := \Psi_* \mu_f. \end{array}$$

An element $f \in L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$ will be called invariant under diffeomorphisms if the corresponding measure μ_f has this property, i.e. if $\Psi_* \mu_f = \mu_f$, for every $\Psi \in \mathrm{Diff}_0(M)$.

Finally consider again the family of functionals $\{\varphi_\Gamma\}(\Gamma \in L)$ of the previous section and define the so-called **covariance condition** as:

$$\varphi_{\Gamma'} \circ \Psi_* = \varphi_\Gamma \qquad \text{whenever } \varphi(\Gamma) = \Gamma'.$$

As the same, one says that the family of probability measures $\{\mu_\Gamma\}(\Gamma \in L)$ satisfies the covariance condition if the corresponding family of functionals does.

It is straightforward to see that if the condition of covariance is satisfied by a promeasure, then the cylindrical measure induced on $\overline{\mathcal{A}/\mathcal{G}}$ is invariant under diffeomorphisms. This assertion is formalized in the next theorem.

Theorem 5.3.2 *There is a bijection between the diffeomorphism invariant probability measures on $\overline{\mathcal{A}/\mathcal{G}}$ and the probability promeasures $\{\mu_\Gamma\}(\Gamma \in L)$ satisfying the covariance condition.*

I stress that

the uniform measure μ_0 is invariant under diffeomorphisms.

In fact the only possible dependence of μ_0 on the diffeomorphisms of M is contained in the connectivity n_Γ, but this is a topological invariant and so it is unaffected by them, hence the covariance condition is automatically satisfied.

Furthermore note that μ_0 **is also gauge-invariant** since the Haar measures on the compact gauge groups are bi-invariant.

The properties of μ_0 has been widely studied by D.Marolf and J.Mourão in [31]; the most remarkable results obtained in that work are the following:

- μ_0 **is faithful**, i.e. $f \in \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$, $f \geq 0$ and $\int_{\overline{\mathcal{A}/\mathcal{G}}} f \, d\mu_0 = 0$ implies $f \equiv \mathbf{0}$;

- μ_0 **is concentrated on the generalized connections**, i.e.

$$\begin{cases} \mu_0(\mathcal{A}/\mathcal{G}) = 0; \\ \mu_0(\overline{\mathcal{A}/\mathcal{G}}) = 1. \end{cases}$$

5.4 The algorithm of the loop quantization

The loop quantization is a program of **canonical and non-perturbative quantization** of gauge theories. This program is quite recent and it is still far from having a complete formulation, in particular at a dynamical level.

Two standard references about the loop quantization are the book [18] and the lecture note [40].

The first important assumption in the program relies in the choice of a manifestly gauge-invariant description of gauge theories: the configuration space of these theories is taken to be $\mathcal{A}/\mathcal{G}$ and not $\mathcal{A}$.

With this choice the invariance under gauge transformations is solved at a classical level, being encoded in the configuration space itself, and it doesn't need to be implemented as a constraint in the quantum theory.

However this advantage is paid by a remarkable complication in the mathematical structure: $\mathcal{A}/\mathcal{G}$ is a non-flat space with a highly non-trivial topology, for many interesting theories it *isn't* a manifold and, most important, there are several obstructions to construct measures on it.

These problems explain why, in the usual Hamiltonian quantization of gauge theories, the configuration space is always taken to be $\mathcal{A}$. The invariance under gauge transformations is then introduced as a constraint (**the Gauss constraint**) by means of several procedures as ghosts, gauge fixing, projections and so on, see [22] for a more complete discussion on these topics.

Nevertheless the lattice gauge theory suggested a way to avoid these problems by using the techniques related to the Wilson functions, this suggestion is sensed because, as shown in chapter 4, the gauge invariant information of the connections is fully encoded in the Wilson functions, hence one is naturally leaded to assume **the holonomy algebra as the algebra of the classical observables of gauge theories**.

The quantization is then performed by means of the C^*-algebraic formalism: the self-adjoint elements of the holonomy algebra are promoted to self-adjoint linear operators on a certain Hilbert space containing the **kinematical states** of the quantum theory.

About this Hilbert space it is useful to remember that what usually happens in the quantization of gauge theories (see [20] or [31]) is that on the classical configuration space, denoted generically with X, there is a cylindrical but not σ-additive measure μ which enables to construct the pre-Hilbert space $L^2_{cyl}(X, \mu)$ of the square-integrable cylindrical functions on X; if μ admits an extension to a Borel measure $\bar{\mu}$ on X then the completion of $L^2_{cyl}(X, \mu)$ leads to the Hilbert space $L^2(X, \bar{\mu})$.

However, if this extension is not available, the quantum theory is implemented by extending (on the base of physical and/or mathematical considerations) the classical configuration space X to a wider space $\bar{X}$ on which a genuine measure ν is available, in order to construct the Hilbert space $L^2(\bar{X}, \nu)$.

The space $\bar{X}$ is called **the quantum configuration space** and the Hilbert space $L^2(\bar{X}, \nu)$ is taken to be **the space of the quantum kinematical states** of the theory.

This is precisely what happens in the loop quantization of gauge theories: the lack of a measure on $\mathcal{A}/\mathcal{G}$ leads to search an extension of this space, the major candidate to the role of quantum configuration space is $\overline{\mathcal{A}/\mathcal{G}}$ for the following reasons:

- first of all $\mathcal{A}/\mathcal{G}$ is injectively and densely embedded in $\overline{\mathcal{A}/\mathcal{G}}$, hence the classical theory is contained in the quantum theory without anomalies;

- $\overline{\mathcal{A}/\mathcal{G}}$ is an infinite-dimensional compact Hausdorff space endowed with a *natural* probability measure, the uniform measure μ_0. Associated to this (faithful) measure there is one and only one faithful representation of the holonomy C^*-algebra $Hol(M, G)$ supported by the Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$, i.e. the GNS representation:

$$\begin{array}{ccc} Hol(M,G) & \longrightarrow & \mathcal{B}(L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)) \\ f & \mapsto & M_{\hat{f}} \end{array}$$

  $M_{\hat{f}}(\psi) := \hat{f}(\bar{A})\psi(\bar{A})$, $\forall \psi \in L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$, $\hat{f} \in \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}}) \subset L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ is the Gelfand transformed of f. Hence the elements of the holonomy C^*-algebra are promoted to bounded multiplication operators on the Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$, they are bounded because the Wilson functions (which generate $Hol(M, G)$) are bounded and the Gelfand isomorphism is isometric. The real part of the Wilson functions are thus promoted to bounded self-adjoint operators on $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$, i.e. observables in the quantum theory. It is worth noting that the real parts of the Wilson functions generate the same C^*-algebra thanks to the identity ${T_\alpha}^* = T_{\alpha^{-1}}$;

- while the previous are mathematically rigorous motivations for the choice of $\overline{\mathcal{A}/\mathcal{G}}$ as the quantum configuration space, there is a further motivation based on a physical intuition. Since, as proved in chapter 3, gauge-equivalence connections are in one-to-one correspondence to conjugation classes of holonomies, in a lattice gauge theory based on a graph Γ, the configuration space is G^{n_Γ}/Ad_G, hence, being $\overline{\mathcal{A}/\mathcal{G}}$ the projective limit of the family $\{G^{n_\Gamma}/Ad_G\}_\Gamma$, a gauge field theory which has $\overline{\mathcal{A}/\mathcal{G}}$ as quantum configuration space is suitable to be interpreted as the continuous limit of the lattice gauge theories corresponding to every fixed graph, which are approximated (or regularized) theories. The fact that the set of all graphs is closed under diffeomorphisms is of essential importance when the diffeomorphism invariance is taken into account. For the reasons discussed above, a graph Γ is interpreted in the formalism of the loop quantization as a *floating lattice in* M.

The compactification of the configuration space is not a characteristic of this procedure, but it often appears in the quantization of the systems with an infinite number of degrees of freedom, such as field theories. For example in the quantization of the scalar field in d-dimensions the classical configuration space, i.e. the Schwartz space $S(\mathbb{R}^d)$, is substituted by $S'(\mathbb{R}^d)$, the space of the tempered distributions on $\mathbb{R}^d$, in which it is densely embedded.

I stress that the compactification $\mathcal{A}/\mathcal{G} \hookrightarrow \overline{\mathcal{A}/\mathcal{G}}$ is highly non-trivial, since the uniform measure μ_0 restricted to $\mathcal{A}/\mathcal{G}$ is the null measure. This fact has put in evidence the important role of the generalized connections in the loop quantization.

The previous (both physical and mathematical) considerations, can be summarized in the steps of the algorithm of the loop quantization (at a kinematical level).

The algorithm of the loop quantization:

1. the configuration space of a given classical gauge theory is taken to be $\mathcal{A}/\mathcal{G}$;

2. the algebra of the classical configuration observables of the theory is assumed to be the holonomy C^*-algebra $Hol(M, G)$;

3. **axioms of quantization**: the quantum theory (at the kinematical level) is implemented by the GNS representation of $Hol(M, G)$ supported by Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$.

As said before, the first two assumptions solve already at the classical level the Gauss constraint generated by the invariance under gauge transformations.

If the gauge theory is also invariant under diffeomorphisms (as the general relativity in Ashtekar's formulation), then the constraint generated by this new kind of invariance is imposed at the quantum level by selecting a suitable subspace of the kinematical state space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ given by the states satisfying this constraint.

In the canonical quantization one operates the splitting of the space-time in space+time, hence there are two kind of constraints generated by the invariance under diffeomorphism: the constraint depending on the spatial part and that depending on the temporal evolution, called **Hamiltonian constraint**.

There is an important instrument to construct spatial diffeomorphism invariant states called **loop transform**, this is a linear operator enables to pass from the quantum representation described above, called **connection representation** (because the states are functions of generalized connections), to the so-called **loop representation**, where the states are functions of loops.

The importance of the loop representation is that this representation carries topological invariants, exhibited by the loop transform and called **generalized knot-invariants**, which naturally satisfy the diffeomorphism constraint.

To the description of the loop representation is entirely dedicated the following chapter.

Chapter 6

The loop representation of diffeomorphism invariant gauge theories

It is well known that, in ordinary quantum mechanics, the Fourier transform enables to pass from the **position representation**, in which the states are functions of the generalized coordinates q^i, to the **momentum representation**, in which the states are functions of the momentum coordinates p_j.

The Fourier transform is a unitary operator from $L^2(\mathbb{R}^3)$ into itself, thus the momentum and the position representation are two physically equivalent description of the quantum 3-D world since unitary operators preserve the scalar brackets and then the expectation values of the observables.

The usefulness of the unitary correspondence between 'position states' and 'momentum states' induced by the Fourier transform is due to the fact that in many situations of physical and mathematical interest the equations of quantum mechanics are much more easily solved in the momentum representation.

Starting from this consideration Rovelli and Smolin proposed in [38] a formal transform, called **loop transform** and deeply related to the Fourier transform, to pass from the 'connection representation', in which the states are the normalized vectors $|\psi> \in L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$, to a 'loop representation', in which the states are functions of loops.

The advantage is that it is easier to find out diffeomorphism invariant functions of loops instead of diffeomorphism invariant functions of connections.

The proposal of Rovelli and Smolin was only formal in 1990, but after the construction of the uniform measure on $\overline{\mathcal{A}/\mathcal{G}}$ the expression of the loop transform can be written rigorously as

$$L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0) \ni \psi \rightsquigarrow \ell_\psi(\alpha) := \int_{\overline{\mathcal{A}/\mathcal{G}}} \overline{T_\alpha(\bar{A})}\psi(\bar{A})d\mu_0(\bar{A})$$

well defined after one identifies the Wilson function T_α with its Gelfand transform, which belongs to $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}}) \subset L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$.

As a consequence of the diffeomorphism invariance of μ_0, if ϕ is an orientation preserving diffeomorphism of M, then $\ell_\psi(\alpha) = \ell_\psi(\phi \circ \alpha)$, i.e. ℓ_ψ assumes the same value on every loop obtained by a fixed loop through a diffeomorphic deformation. Thus the states ℓ_ψ satisfy the diffeomorphism invariance in a natural fashion and hence are good candidates to be 'loop states' of the quantum gravity.

Three questions naturally arise:

1. what is the domain of the functions ℓ_ψ?

2. what is the structure of the set $\{\ell_\psi \mid \psi \in L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)\}$?

3. what are the properties of the loop transform? In particular, is it a unitary operator?

Observe that the domain of the functions ℓ_ψ is the **configuration space in the loop representation** and the Hilbert completion of the set of these functions is the **kinematical state space of the loop representation**.

The questions above are of fundamental importance, in fact, as remarked above, the connection representation of quantum gravity and the loop representation are physically equivalent (i.e. give the same expectation values of the observables) if and only if the transform which relates the two representations, i.e. the loop transform, is unitary!

The full proof of the unitary character of the loop transform written in the form above is still under investigation, but, as will be shown later, there is a complete and rigorous theory of the loop transform for Abelian quantum gauge theories (e.g. quantum electrodynamics) in which the loop transform is proved to be a genuine unitary operator between Hilbert spaces.

At the time of writing, the more promising and useful way to construct a 'loop representation' is given by the theory of **spin network**, the discussion of which is beyond the scope of this paper. The interested reader can find a complete description of the loop representation in terms of spin-networks in [40].

6.1 The early days of the loop transform

The first results on the loop transform have been obtained by Ashtekar and Isham in [4].

To describe these results it is worth introducing the following symbols:

- $\mathcal{W} := \{T_\alpha \mid \alpha \in L_\star(M)\}$;
- $\mathcal{P}(\mathcal{W})$ is the set of the complex-valued positive-definite functions on $\mathcal{W}$, where a complex valued function ℓ on $\mathcal{W}$ is said to be positive-definite if
$$\sum_{i=1}^{n} c_i \ell(T_{\alpha_i}) \geq 0 \text{ whenever } \sum_{i=1}^{n} c_i T_{\alpha_i}(\bar{A}) \geq 0$$
for every $\bar{A} \in \overline{\mathcal{A}/\mathcal{G}}$ and with $c_i \in \mathbb{C}$;
- $M_+(\overline{\mathcal{A}/\mathcal{G}})$ is the set of the regular Borel positive measures on $\overline{\mathcal{A}/\mathcal{G}}$.

The most important result obtained by Ashtekar and Isham is a sort of generalization of Bochner's theorem of harmonic analysis.

Theorem 6.1.1 *If the gauge group G is $U(1)$ or $SU(2)$ then the map*
$$\begin{array}{rccc} \mathcal{L}: & M_+(\overline{\mathcal{A}/\mathcal{G}}) & \longrightarrow & \mathcal{P}(\mathcal{W}) \\ & \mu & \mapsto & \mathcal{L}(\mu) := \ell_\mu \end{array}$$
where $\ell_\mu(T_\alpha) := \int_{\overline{\mathcal{A}/\mathcal{G}}} \overline{T_\alpha} d\mu$, is a bijection.

For a rigorous proof of this theorem the interest reader is referred to [2].

If $\mu \in M_+(\overline{\mathcal{A}/\mathcal{G}})$, then every element $\psi \in L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu) \subset L^1(\overline{\mathcal{A}/\mathcal{G}}, \mu)$ is univocally associated (through the Radon-Nikodym theorem) to a measure μ_ψ in $M(\overline{\mathcal{A}/\mathcal{G}})$ such that $\int_{\overline{\mathcal{A}/\mathcal{G}}} \overline{T_\alpha} d\mu_\psi = \int_{\overline{\mathcal{A}/\mathcal{G}}} \overline{T_\alpha} \psi d\mu$.

Hence the loop transform can be obtained simply by restriction of $\mathcal{L}$ to $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$ for a suitable μ. This map is an injective linear operator and it is also continuous w.r.t. the L^2 norm because $|\ell_{\mu_\psi}(T_\alpha)| \leq \|\psi\|_2 \|\mu\|^{\frac{1}{2}}$.

It is possible to extend $\mathcal{L}$ to a unitary operator between Hilbert spaces as follows:

- introduce a scalar product in $Im(\mathcal{L})$ in this way:
$$\forall \ell_\psi, \ell_\varphi \in Im(\mathcal{L}) \qquad (\ell_\psi \mid \ell_\varphi) := (\mathcal{L}^{-1}\ell_\psi \mid \mathcal{L}^{-1}\ell_\varphi) \equiv (\psi \mid \varphi)$$
- complete $Im(\mathcal{L})$ to a Hilbert space in the norm generated by this scalar product and indicate it as $\overline{Im\mathcal{L}}$;

- finally extend $\mathcal{L}$ to a unitary operator between $L^2(\overline{\mathcal{A}/\mathcal{G}},\mu)$ and $\overline{Im\mathcal{L}}$ through:

$$\begin{array}{rcl} L^2(\overline{\mathcal{A}/\mathcal{G}},\mu) & \to & \overline{Im\mathcal{L}} \\ \psi & \mapsto & \ell_\psi \end{array}$$

where ℓ_ψ is the function:

$$\begin{array}{rrcl} \ell_\psi : & \mathcal{W} & \to & \mathbb{C} \\ & T_\alpha & \rightsquigarrow & \ell_\psi(T_\alpha) := \lim_n \ell_{\psi_n}(\alpha) \end{array}$$

where $\psi_n \to \psi$ as $n \to +\infty$.

The main lacks of this construction are that

1. the range of the loop transform constructed in this way, i.e. the kinematical quantum state space of the loop representation, is not characterized in an explicit fashion;

2. the construction works only when G is $U(1)$ or $SU(2)$. The reason why this happens relies in the fact that these groups are the only subgroups of $SU(N)$ for which the linear span of the Wilson functions agrees with the algebra they generate and this circumstance is essential to prove the theorem.

6.2 The inductive construction of the loop transform

There is a different way to construct the loop transform of Rovelli and Smolin in a rigorous way, this construction uses inductive techniques and has the great advantage w.r.t. that presented in the previous section to show the structure of the kinematical state space of the loop representation explicitly.

This inductive construction of the loop transform leads to a full control of the correspondence between connection and loop representation, but it is, at the moment, available only in the Abelian case, i.e. when $G = U(1)$.

The construction begins by constructing $\overline{\mathcal{A}/\mathcal{G}}$ as a projective limit of tori following [31]. Let us consider the set J of the subgroups L of $\mathcal{L}_\star(M)$ generated by a finite independent family of loops. By $L \leq L'$ we mean that L is a subgroup of L'; J is directed w.r.t. this ordering.

The projective family associated to $\overline{\mathcal{A}/\mathcal{G}}$ is defined as follows:

- we take as index set the directed set J;

- to every $L \in J$ we associate the group $Hom(L, U(1))$;
- if $L \leq L'$ we define the projection $\pi_{LL'} : Hom(L', U(1)) \to Hom(L, U(1))$ which restricts the homomorphisms $H \in Hom(L', U(1))$ to the subgroup L.

To simplify the notation we denote $Hom(L, U(1))$ by $\overline{\mathcal{A}/\mathcal{G}}_L$ and its dual group[1] by $\mathcal{W}_L$.

For a given independent family of loops $(\alpha_1, ..., \alpha_n)$ the evaluation map

$$ev_{(\alpha_1,...,\alpha_n)}(H) = (H(\alpha_1), ..., H(\alpha_n)) \tag{6.1}$$

is an isomorphism of $\overline{\mathcal{A}/\mathcal{G}}_L$ with the n-dimensional torus $U(1)^n$.

The group $\overline{\mathcal{A}/\mathcal{G}} \equiv Hom(\mathcal{L}_\star(M), U(1))$ is the projective limit of this family and the projection $\pi_L : \overline{\mathcal{A}/\mathcal{G}} \to \overline{\mathcal{A}/\mathcal{G}}_L$, which restricts the homomorphisms $H \in Hom(\mathcal{L}_\star(M), U(1))$ to L, is continuous and surjective owing to the interpolation property of independent loops.

The loop transform $\mathcal{L}$ will be constructed as the inductive limit of the Fourier transforms F_L between the Hilbert spaces $L^2(\overline{\mathcal{A}/\mathcal{G}}_L)$ and $L^2(\mathcal{W}_L)$ (for shortness the relative Haar measures have been omitted).

The scheme of the work is visualized in this diagram:

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
L^2(\overline{\mathcal{A}/\mathcal{G}}_L) & \stackrel{F_L}{\longrightarrow} & L^2(\mathcal{W}_L) \\
\downarrow i_{L'L} & \vdots & \downarrow j_{L'L} \\
L^2(\overline{\mathcal{A}/\mathcal{G}}_{L'}) & \stackrel{F_{L'}}{\longrightarrow} & L^2(\mathcal{W}_{L'}) \\
\vdots & \vdots & \vdots \\
\downarrow & \downarrow & \downarrow \\
L^2(\overline{\mathcal{A}/\mathcal{G}}) & \stackrel{\mathcal{L}}{\longrightarrow} & L^2(\mathcal{W}) \equiv L^2(Hoop_\star(M))
\end{array}$$

where $\mathcal{W} = \varinjlim_L \mathcal{W}_L$ can be proved to agree with $\mathcal{H}_\star(M, U(1))$.

To make the family $\{L^2(\overline{\mathcal{A}/\mathcal{G}}_L)\}$ an inductive family of Hilbert spaces we define the inclusions $i_{L'L}$ for $L \leq L'$ as follows: for every $\psi \in L^2(\overline{\mathcal{A}/\mathcal{G}}_L)$ we put

$$(i_{L'L}\psi)(H') := \psi(\pi_{LL'}(H')) \qquad H' \in \overline{\mathcal{A}/\mathcal{G}}_{L'}.$$

These inclusions are linear and satisfy the consistency conditions, so we have only to prove that they are isometric maps. Suppose that L and L'

[1]The dual group of a locally compact group is the Abelian group of its continuous characters, i.e. the homomorphism from the group to $U(1)$, it is itself a locally compact group in the topology of the uniform convergence on the compact subsets.

are the free groups generated by the independent families $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_{n'}\}$, respectively, and that $L \leq L'$. We have:

$$\begin{cases} \alpha_1 = {\beta_1}^{k_{1,1}} \dots {\beta_{n'}}^{k_{n',1}} \\ \vdots \\ \alpha_n = {\beta_1}^{k_{1,n}} \dots {\beta_{n'}}^{k_{n',n}} \end{cases}$$

for some $k_{r,s} \in \mathbb{Z}$ for $r = 1, \dots, n'$ and $s = 1, \dots, n$.

For $ev_{(\beta_1,\dots,\beta_{n'})}(H) = (e^{i\vartheta_1}, \dots, e^{i\vartheta_{n'}})$, it follows

$$ev_{(\alpha_1,\dots,\alpha_n)}(\pi_{L'L}(H)) = (e^{ik_{1,1}\vartheta_1} \cdots e^{ik_{n',1}\vartheta_{n'}}, \dots, e^{ik_{1,n}\vartheta_1} \cdots e^{ik_{n',n}\vartheta_{n'}}) \,.$$

By composition of the evaluation maps with $i_{L'L}$ one obtains the inclusions $i_{n'n} : L^2(U(1)^n) \to L^2(U(1)^{n'})$, defined by

$$(i_{n'n}\psi)(e^{i\vartheta_1}, \dots, e^{i\vartheta_{n'}}) = \psi(e^{i(k_{1,1}\vartheta_1 + \dots + k_{n',1}\vartheta_{n'})}, \dots, e^{i(k_{1,n}\vartheta_1 + \dots + k_{n',n}\vartheta_{n'})}).$$

From the normalization and the bi-invariance of the Haar measure it follows that the inclusions $i_{n'n}$, and hence also the inclusions $i_{L'L}$, are isometric.

The inclusions $j_{L'L}$ are defined by the following commutative diagram:

$$\begin{array}{ccc} L^2(\overline{\mathcal{A}/\mathcal{G}}_{L'}) & \xrightarrow{F_{L'}} & L^2(\mathcal{W}_{L'}) \\ \uparrow i_{L'L} & & \uparrow j_{L'L} \\ L^2(\overline{\mathcal{A}/\mathcal{G}}_{L}) & \xrightarrow{F_{L}} & L^2(\mathcal{W}_{L}) \end{array}$$

They are isometries as compositions of isometric maps. The diagram shows that the consistency condition holds both for the inclusions $j_{L'L}$ and the Fourier transforms F_L. So we have well defined inductive families.

Theorem 6.2.1 *The following assertions hold:*

1. $L^2(\overline{\mathcal{A}/\mathcal{G}})$ *is the inductive limit of* $\{L^2(\overline{\mathcal{A}/\mathcal{G}}_L)\}_L$;
2. $L^2(\mathcal{W})$ *is the inductive limit of* $\{L^2(\mathcal{W}_L)\}_L$;
3. *the loop transform* $\mathcal{L}$ *on* $L^2(\overline{\mathcal{A}/\mathcal{G}})$ *is the inductive limit of* $\{F_L\}_L$.

Proof. Let us define the inclusions $i_L : L^2(\overline{\mathcal{A}/\mathcal{G}}_L) \to L^2(\overline{\mathcal{A}/\mathcal{G}})$ by $(i_L\psi_L)(H) = \psi_L(\pi_L(H))$, $\psi_L \in L^2(\overline{\mathcal{A}/\mathcal{G}}_L)$.

Denoting by μ_L the Haar measure on $\overline{\mathcal{A}/\mathcal{G}}_L$ and by μ_0 the normalized Haar measure on $\overline{\mathcal{A}/\mathcal{G}}$, we have that $(\pi_L)_*\mu_0 = \mu_L$. Therefore

$$\|i_L\psi_L\|^2 = \int_{\overline{\mathcal{A}/\mathcal{G}}} \mid \psi_L \circ \pi_L \mid^2 d\mu_0(\bar{A}) = \int_{\overline{\mathcal{A}/\mathcal{G}}_L} \mid \psi_L \mid^2 d\mu_L = \|\psi_L\|^2$$

so that the inclusions i_L are isometric. Moreover their images contain the Wilson functions, hence they cover a dense linear subspace of $L^2(\overline{\mathcal{A}/\mathcal{G}})$. We conclude that $L^2(\overline{\mathcal{A}/\mathcal{G}})$ is the inductive limit of the family $\{L^2(\overline{\mathcal{A}/\mathcal{G}_L})\}_L$.

Now we define the inclusions $j_L : L^2(\mathcal{W}_L) \to L^2(\mathcal{H}_\star(M, U(1)))$ by the following commutative diagram:

$$\begin{array}{ccc} L^2(\overline{\mathcal{A}/\mathcal{G}}) & \xrightarrow{\mathcal{L}} & L^2(\mathcal{H}_\star(M, U(1))) \\ \uparrow_{i_L} & & \uparrow_{j_L} \\ L^2(\overline{\mathcal{A}/\mathcal{G}_L}) & \xrightarrow{F_L} & L^2(\mathcal{W}_L) \end{array}$$

Repeating the same arguments on the inclusions j_L we get that $L^2(\mathcal{H}_\star(M, U(1)))$ is the inductive limit of the family $\{L^2(\mathcal{W}_L)\}_L$ and that the inductive limit of $\{F_L\}_L$ is the loop transform on $L^2(\overline{\mathcal{A}/\mathcal{G}})$. □

There is a hope to extend the construction just shown to the non-Abelian case by replacing the Fourier transforms with the Peter-Weyl transforms, this possibility is still under analysis.

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