Generic expansions of countable models

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Abstract

We compare two different notions of generic expansions of countable saturated structures. On one hand there is a kind of genericity related to model-companions and to amalgamation constructions à la Hrushovski-Fraïssé; on the other, there is a notion of generic expansions defined via topological properties and Baire category theory. The second type of genericity was first formulated by Truss for automorphisms. We work with a later generalization, due to Ivanov, to finite tuples of predicates and functions.

Let N be a countable saturated model of some complete theory T, and let (N, σ) denote an expansion of N to the signature L_0 which is a model of some universal theory T_0. Let $T_{\rm rich}$ be the theory of some suitably defined generic (or *rich*, in our terminology) model, obtained from the amalgamation class naturally associated to T_0. We prove that (N, σ) is Truss-generic if and only if (N, σ) is an e-atomic model. When T is ω–categorical and $T_{\rm rich}$ is model-complete the e-atomic models are simply the atomic models of $T_{\rm rich}$.

1 Introduction

In model theory and descriptive set theory there are two main notions of a *generic expansion* of a model. In some cases, the expansions of a given model that one obtains through these notions are similar enough that it is natural to ask whether, and how, they are related.

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Let T be a theory with quantifier elimination in a language L. Let $L_0 = L \cup \{f\}$, where f is a unary function symbol. Let T_0 be T together with the sentences which say that f is an automorphism.

One notion of genericity was introduced by Lascar in his seminal paper [Lasc2]. Lascar constructs some models of T_0 that have certain properties of universality and homogeneity. The interpretations of f in these models are called *beaux automorphismes* in [Lasc2], and *generic* automorphisms later on (e.g. [ChaHr] and [ChaPi]). When T_0 has a model companion T_c, T_c turns out to be the theory of these universal homogeneous models and, in this case, all sufficiently saturated models of T_c are generic automorphisms (see [ChaPi]).

A second notion of genericity was introduced by Truss in [Tru1]. The interpretation of f in a countable model $M \models T_0$ is Truss-generic if its conjugacy class is comeagre in the canonical topology on $\mathrm{Aut}(M)$. More generally, a tuple $(f_1, \ldots, f_n) \in \mathrm{Aut}(M)^n$ is generic in this sense if $\{(f_1^g, \ldots, f_n^g) : g \in \mathrm{Aut}(M)\}$ is comeagre in the product space $\mathrm{Aut}(M)^n$. Truss-generic automorphisms populate rather different habitats: they are a useful tool in the two main techniques for reconstructing ω–categorical structures from their automorphism group, namely, the small index property [Lasc1] and Rubin's weak $\forall\exists$-interpretations [Rubin] (see e.g. [HHLS] and [BaMac] for specific applications of Truss generics). The existence of a comeagre conjugacy class is often interesting in its own right: for an ω-categorical structure M, it implies that $\mathrm{Aut}(M)$ cannot be written non trivially as a free product with amalgamation [MacTh]. See also a recent paper by Kechris and Rosendal [KeRo] for a wealth of topological consequences in Polish groups.

Now, let us slightly modify the setting: T and L are as above, but L_0 is an expansion of L by a relation symbol r and T_0 is an expansion of T by universal axioms.

Ivanov generalises Truss-genericity so that it applies to predicates, and indeed to arbitrary finite signatures [Ivan]. His work concerns the relation of 'generic expansions' of ω-categorical structures to generalized quantifiers in the context of second-order logic. Lascar's genericity also applies to predicates: in [ChaPi] the authors show that for a complete theory T, $L_0 = L \cup \{r\}$, where r is a unary relation and $T_0 = T$, T_0 has a model companion if and only if T eliminates the $\exists^\infty$ quantifier.

Models obtained by a generalised version of the Fraïssé construction yield yet another kind of genericity. In most cases, this is known to be closely related to the first notion mentioned above. Many examples can be found among ω–categorical structures. For instance, the random graph can be defined either as a Fraïssé limit or as the model companion of the theory T_0 of an irreflexive symmetric binary relation on an infinite set. For uniformity with the previous examples, we may think of T_0 as an expansion of the theory of an infinite set in the empty language L. Another example is Hrushovski's new strongly minimal set [Hru1] (the famous construction which refutes Zilber's trichotomy conjecture): this structure can be thought of as the model companion of some cleverly chosen theory T_0 in the language L_0 that only contains a ternary predicate.

In Sections 2 and 3 we introduce a minor generalization of Fraïssé constructions: we consider amalgamation classes where the joint embedding property (JEP) fails, and we call them *unconnected*. The motivation is to provide a framework where the two definitions of genericity can be compared, by abstracting the properties that yield the relevant results in [Lasc2] and [ChaPi]. In Sections 4 and 5 we clarify the relation between generic models obtained through generalised Fraïssé constructions (sometimes called *Hrushovski amalgamations*) and generic models defined using the model companion of some theory. They are equivalent when the amalgamation class involved is *full* (Definition 4.6, Theorem 5.3). The hypothesis of fullness is necessary by Remark 5.4 but it holds trivially when the amalgamation class is *connected* (i.e. JEP holds).

In Sections 6 and 7 we work with a given countable saturated model $N \models T$ and we consider the set $\mathrm{Exp}(N, T_0)$ of expansions of N that model T_0. We endow $\mathrm{Exp}(N, T_0)$ with a topology which makes it a Polish space. Our topology is the one in [Ivan], a natural generalisation of the canonical topology on $\mathrm{Aut}(N)$. Let T_{rich} be the theory of the expansions of N that are generic in the sense of generalised Fraïssé constructions. We prove that the existentially closed expansions of N are a comeagre subset of $\mathrm{Exp}(N, T_0)$.

In Section 6 we also define a set of 'slighly saturated' expansions of N which we call *smooth*. A smooth expansion of N realizes all types of the form $(*)$ $p_{\restriction L}(x) \cup \{\varphi(x)\}$, where $p_{\restriction L}(x)$ is a type in the base language L and $\varphi(x)$ is a quantifier-free formula in the expanded language L_0. We prove that smooth expansions are a comeagre subset of $\mathrm{Exp}(N, T_0)$. Finally, in Section 7 we define *e-atomic* expansions. An e-atomic expansion is existentially closed, smooth, and only realizes $p(x)$ if $p_{\restriction\forall}(x) \cup p_{\restriction\exists}(x)$ is isolated by types of the form $\exists y\, p(x, y)$, where $p(x, y)$ is as in $(*)$. We show that e-atomic expansions are exactly the expansions that are generic in the sense of [Tru1]. When T is ω-categorical and T_{rich} is model-complete, this amounts to showing that Truss generic expansions are exactly the atomic models of T_{rich}.

Finally, in Section 8 we give some examples: Truss-generic automorphims of the random graph and Truss-generic black fields.

When Σ is a (possibly infinite) set of formulas, we shall sometimes write $\Sigma + \varphi$ to mean $\Sigma \cup \{\varphi\}$, and $\Sigma \to \varphi$ for $\Sigma \vdash \varphi$.

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2 Inductive amalgamation classes.

In this section we give a self-contained axiomatization of *inductive amalgamation classes*. In a terminology which has recently gained popularity, we axiomatize a type of *abstract elementary class (AEC)*, see e.g. [Bald]. The AECs we consider live in first-order logic: in our setting, the relation $\preceq_{\mathcal{K}}$ contains the relation of elementary substructure in some first-order language. There is a stylistic difference in our approach: we base the axiomatization on the notion of morphism, while in the literature on AEC the primitive notion is that of submodel (here denoted by $\leq$, elsewhere often denoted by $\preceq_{\mathcal{K}}$).

An inductive amalgamation class contains structures and partial maps between structures. It is partitioned into connected components, within each of which an appropriate version of the joint embedding property holds. The trivial, yet canonical, example contains all the infinite models of a given theory T_0 and all partial elementary maps between models. This class is not connected unless T_0 is complete. The connected components consist of models of the same completion of T_0. The saturated models are the *rich* models of this class, according to the terminology defined below.

A concrete example of an inductive amalgamation class is given by fields and partial isomorphisms (i.e. restrictions of isomorphisms between subrings). This class is not connected: each connected component contains all the fields of a given characteristic. Algebraically closed fields with infinite transcendence degree are the rich models of this class.

Finally, highly non trivial examples are obtained from Hrushovski-style constructions such as [Hru1]: in such settings, one works with an inductive amalgamation class where structures are the models of some theory T_0 and the maps considered are partial isomorphisms between self-sufficient subsets.

Throughout this paper, a **map** is a triple $f : M \to N$ where M is a structure called the **domain** of the map, N is a structure called the **co-domain** of the map, and

f is a function in the set-theoretic sense with $\operatorname{dom} f \subseteq M$ and $\operatorname{rng} f \subseteq N$. We call $\operatorname{dom} f$ the **domain of definition** of the map and $\operatorname{rng} f$ the **range** of the map. If $A \subseteq \operatorname{dom} f$ we say that f is **defined on** A. The **composition** of two maps is defined in the obvious way when the co-domain of the first map is the domain of the second map. The **inverse** of $f : M \to N$ is defined when f is injective (which will always be the case in this paper) and it is the map $f^{-1} : N \to M$.

Fix a countable language L and let M and N be two structures of signature L. A **partial isomorphism** is a map $f : M \to N$ such that

$$M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(fa)$$

for every quantifier-free formula $\varphi(x)$ and every tuple $a \in \operatorname{dom} f$. In other words, f can be extended to an isomorphism between substructures. An **elementary map** is defined similarly but with $\varphi(x)$ ranging over all formulas. A partial isomorphism which is a total map is called **quantifier-free embedding** and a total elementary map is called an **elementary embedding**. Generally, in the literature, the term *embedding* tout court means *quantifier-free embedding*, but in our context it is more natural to use *embedding* as in Definition 2.1 below.

An **amalgamation class** is a nonempty class $\mathcal{K}$ containing infinite[1] structures of signature L and maps between structures that satisfy axioms K0-K4, R and AP below. Henceforth, by **model** we shall mean "structure in $\mathcal{K}$", and by **morphism** "map in $\mathcal{K}$". The letters $\boldsymbol{M}$, $\boldsymbol{N}$, etc. will denote models. All the notions introduced below are relative to $\mathcal{K}$, though reference to it is usually omitted.

K0 The class of models is closed under elementary equivalence.

K1 Every morphism is a partial isomorphism.

K2 Every elementary map is a morphism.

K3 The class of morphisms is closed under inverse and composition.

We shall write $\boldsymbol{M \leq N}$ when $M \subseteq N$ and $\mathrm{id}_M : M \to N$ is a morphism. We say that M is a **submodel** of N, or that N **extends** M. By K2, submodels are substructures and elementary substructures are submodels. From K3 it follows that $\leq$ is a transitive relation. Moreover, if $K \subseteq N \leq M$, then $K \leq M$ iff $K \leq N$. In fact, if $K \leq M$ then by transitivity $K \leq N$. For the converse it suffices to compose

[1]Traditionally, the amalgamation classes used in Fraïssé-like constructions only contain finite structures, which are not apt to describe other recent applications (e.g. Examples 6.9 and 8.4). Infinite structures offer a more convenient framework. We do not have a general recipe for translating finite amalgamation classes into inductive amalgamation classes: an ad hoc approach in many examples, though, suggests that such a translation is always possible (cf. Examples 3.2 and 8.1).

$\mathrm{id}_K : K \to N$ with the inverse of $\mathrm{id}_M : M \to N$.

2.1 DEFINITION. A total morphism $f : M \to N$ is called an **embedding** of M into N. □

If $f : M \to N$ is en embedding then $f[M] \leq N$. In fact, as $f^{-1} : f[M] \to M$ is an isomorphism, by K2 it is a morphism. By composing it with $f : M \to N$, we can conclude that the natural embedding of $f[M]$ into N is also a morphism.

It will be convenient to use the following axiom, which is rather harmless.

R The restriction of a morphisms is a morphism. That is if $f : M \to N$ is a morphism and $h \subseteq f$ then $h : M \to N$ is also a morphism.

The following axiom is called **amalgamation property**:

AP If $f_i : M \to N_i$ for $i = 1, 2$ are morphisms then there is a model N and two embeddings $h_i : N_i \to N$ such that $h_1 f_1 = h_2 f_2$.

There is a more compact formulation of the amalgamation property. To state it we need the notion of extension of a morphism: we say that the morphism $f' : M' \to N'$ **extends** $f : M \to N$ if f' extends f as a function, $M \leq M'$, and $N \leq N'$.

AP′ Every morphism $f : M \to N$ has an extension to an embedding $h : M \to N'$.

We claim that AP′ is equivalent to standard amalgamation. To prove that AP′ follows from AP, amalgamate $f : M \to N$ and $\mathrm{id}_M : M \to M$. For the converse direction, extend the map $f_2 f_1^{-1} : N_1 \to N_2$ to an embedding $h : N_1 \to N$. This and $\mathrm{id}_{N_2} : N_2 \to N$ are the two embeddings $h_i : N_i \to N$ required in AP.

We say that the class $\mathcal{K}$ is **connected** if the **joint embedding property** holds, that is:

JEP For every pair of models M_1 and M_2 there are a model N and embeddings $f_i : M_i \to N$ for $i = 1, 2$.

By AP, the relation of being embeddable into the same model is an equivalence relation. We define the **connected components** of $\mathcal{K}$ to be the equivalence classes of this relation.

2.2 REMARK. If in the definition of JEP we replace *embeddings* by *morphisms*, we obtain an equivalent property. By AP, M_i and N embed in some model N_i. Suppose $f_i : N \to N_i$ are the embeddings obtained. A further application of AP to f_1, f_2 gives the required embeddings of each M_i into N'.

2.3 REMARK. As special case of Remark 2.2, we obtain the following: M and N belong to the same connected component iff there is a morphism between M and N.

By a **chain of models** we mean a sequence of models $\langle M_\alpha : \alpha < \lambda \rangle$ such that $M_\alpha \leq M_\beta$ whenever $\alpha < \beta$. An amalgamation class is **inductive** if the following axiom holds.

In The union of a chain of models is a model that extends every element of the chain.

The property that follows is the **finite character** of morphisms. We state it here for reference, but we do not need to assume it as an axiom. In fact, we shall prove it in Theorem 3.7 below.

FC If all finite restrictions of $f : M \to N$ are morphisms then $f : M \to N$ is a morphism.

3 Rich models.

Throughout this section, we shall work in an inductive amalgamation class $\mathcal{K}$.

3.1 Definition. Let λ be an infinite cardinal. A model U is **λ–rich** if every morphism $f : M \to U$ such that $|f| < |M| \leq \lambda$ has an extension to an embedding of M into U. That is, there is a total morphism $h : M \to U$ such that $f \subseteq h$. When $\lambda = |U|$ we say that U is **rich**.

Using the downward Löwenheim-Skolem Theorem and FC, it is not difficult to prove that when λ is uncountable we can replace $|f| < |M| \leq \lambda$ with $|M| < \lambda$ (as in [ChaPi]) and obtain an equivalent notion. On the other hand, for countable λ the situation is different: finite models might not exist, so the notion might be trivial.

3.2 Example. The countable random graph is a rich model of the inductive amalgamation class that contains all infinite graphs and all partial isomorphisms between them. Most Fraïssé limits of finite structures can also be thought of as rich models of a suitably defined inductive amalgamation class.

3.3 Theorem. [Existence] Let λ and κ be cardinals such that $2^\lambda \leq \kappa = \kappa^{<\lambda}$. Then every model U_0 of cardinality $\leq \kappa$ embeds in a λ–rich model U of cardinality κ.

Proof. Let U_0 be given. We may assume $|U_0| = \kappa$. We define by induction a chain of models $\langle U_\alpha : \alpha < \kappa \rangle$ such that $|U_\alpha| = \kappa$ for all $\alpha < \kappa$. Let $U := \bigcup_{\alpha<\kappa} U_\alpha$.

At successor stage $\alpha + 1$, let $f : M \to U_\alpha$ be the least morphism—in a well-ordering that we specify below—such that $|f| < |M| \leq \lambda$ and f has no extension to an embedding $f' : M \to U_\alpha$. Apply AP′ to obtain an embedding $f' : M \to U'$ that extends $f : M \to U_\alpha$. By Löwenheim-Skolem we may assume $|U_\alpha| = |U'|$. Let $U_{\alpha+1} = U'$. At stage with α limit, simply let $U_\alpha := \bigcup_{\beta<\alpha} U_\beta$.

The well-ordering mentioned needs to be chosen so that in the end we forget nobody. So, first at each stage we well-order the isomorphism-type of the morphisms $f : M \to U_\alpha$ such that $f < |M| \leq \lambda$. Then the required well-ordering is obtained by dovetailing all these well-orderings. The length of this enumeration is at most $2^\lambda \cdot \kappa^{<\lambda}$, which is κ by hypothesis.

We check that U is λ–rich. Suppose that $f : M \to U$ is a morphism and $|f| < |M| \leq \lambda$. The cofinality of κ is larger than $|f|$, hence $\operatorname{rng} f \subseteq U_\alpha$ for some $\alpha < \kappa$. So $f : M \to U_\alpha$ is a morphism and at some stage β we have ensured the existence of an extension of $f : M \to U_\alpha$ that embeds M into $U_{\beta+1}$. □

Theorem 3.3 is too general to yield a sharp bound on the cardinality of U. For instance, it cannot be used to infer the existence of countable rich models. However, it will enable us at the end of this section to define T_{rich} for any inductive amalgamation class.

3.4 Corollary. Let λ be an inaccessible cardinal. Then every model of cardinality $\leq \lambda$ embeds in a rich model of cardinality λ. □

We prefer to work with rich, rather than λ-rich, models. In order to guarantee the existence of λ-rich models in full generality, we shall assume the existence of as many inaccessible cardinals as needed. This assumption will make our proofs smoother, but it is not necessary: it could be eliminated at the cost of complicating notation.

3.5 Theorem. [Uniqueness] Let U and V be λ–rich models. Then any morphism $f : U \to V$ is an elementary map. When $|f| < |U| = |V| = \lambda$, f can be extended to an isomorphism.

Proof. We may assume that f is finite. To prove the elementarity of f, it suffices to extend $f : U \to V$ by back-and-forth to an isomorphism between countable elementary substructures of U and V. The details are left to the reader.

To prove the second part of the claim, we extend $f : U \to V$ by back-and-forth, taking care to ensure totality and surjectivity. At limit stages we can safely take unions, since by the first part of the theorem morphisms between U and V are elementary. □

By Remark 2.3 there is a morphism between U and V iff the two models belong to the same connected component. So, given connected component and cardinality there is at most one rich model.

3.6 COROLLARY. [Homogeneity] Rich models are homogeneous in the sense that every morphism $f : U \to U$ of cardinality $< |U|$ has an extension to an automorphism of U. In particular, by K2, rich models are elementarily homogeneous. □

3.7 THEOREM. [Finite character] The map $f : M \to N$ is a morphism iff $h : M \to N$ is a morphism for every finite $h \subseteq f$.

PROOF. One direction is simply axiom R. For the converse, suppose that $h : M \to N$ is a morphism for every finite $h \subseteq f$. By Theorem 3.3 we may assume $M, N \leq U$ for some rich model U. So, $h : U \to U$ is a morphism and, by Theorem 3.5, elementary. So f is also elementary on U, hence it is a morphism by K2. As $M, N \leq U$ the map $f : M \to N$ is also a morphism because it is a composition of morphisms. □

By a **chain of morphisms** we mean a sequence of morphisms $f_\alpha : M_\alpha \to N_\alpha$, where the α–th morphism extends the β-th morphism, for every $\beta < \alpha$. The following is an immediate consequence of the finite character of morphisms.

3.8 COROLLARY. The union of a chain of morphisms is a morphism that extends every element of the chain. □

The following corollary needs to be assumed as an axiom in AECs: in standard axiomatizations such as [BALD] our Axiom K2 is not available, hence Theorem 3.7 does not hold.

3.9 COROLLARY. Let $\langle M_\alpha : \alpha < \lambda \rangle$ be a chain of models. Let $M_\lambda := \bigcup_{\alpha<\lambda} M_\alpha$. If N is a model such that $M_\alpha \leq N$ for every $\alpha < \lambda$ then $M_\lambda \leq N$.

PROOF. By 3.7 and 3.8. □

Note that λ–rich models are in particular ω–rich. So the following corollary of Theorem 3.5 is immediate.

3.10 COROLLARY. In each connected component, all rich models have the same theory and this is also the theory of λ–rich models, for any λ. □

Let $\boldsymbol{T}_{\mathbf{rich}}$ be the set of sentences that hold in every rich model of the class $\mathcal{K}$. This is called the **theory of the rich models**. In general $T_{\rm rich}$ is not a complete theory: it is complete if and only if $\mathcal{K}$ is connected (by Theorem 3.5).

4 Saturation

In this section we show that saturation of rich models is never accidental: rather, it is an intrinsic property of the amalgamation class being considered. This generalizes Proposition 10 in [Lasc2] or also Theorem 2.5 of [KueLa]. We shall also isolate a natural property, which we call *fullness*, and show that it does not hold in general (but it holds trivially in all connected amalgamation classes). In the next section, we shall use this property to obtain another characterization of saturation of rich models.

Throughout this section, we shall work in an inductive amalgamation class $\mathcal{K}$.

4.1 Theorem. Assume that $\mathcal{K}$ is connected. The following are equivalent:
1. some λ–rich model is λ–saturated;
2. all λ–rich models are λ–saturated; and
3. every λ–saturated model $M \vDash T_{\text{rich}}$ is λ–rich.

Proof. We prove $1 \Rightarrow 2$. Let U be a λ–rich and λ–saturated model. Let V be λ–rich. We shall use the fact that every morphism between U and V, or between elementary substructures of them, is an elementary map. This a consequence of Theorem 3.5. Let $a \in V$ be a tuple of length $< \lambda$. Let x be a finite tuple of variables. We claim that any type $p(x, a)$ is realized in V. Let V' be a model of cardinality $\leq \lambda$ such that $a \in V' \preceq V$. Since $\mathcal{K}$ is connected there is an elementary embedding $f : V' \to U$. Let c be such that $U \vDash p(c, fa)$. Let U' be a model of cardinality $\leq \lambda$ such that $fa, c \in U' \preceq U$. Let $h : U' \to V$ be an elementary embedding that extends $f^{-1} : U' \preceq V$. Then hc is the required realisation of $p(x, a)$ in V.

To prove $2 \Rightarrow 3$, assume that M is a λ–saturated model such that $M \vDash T_{\text{rich}}$. Let U be a λ–rich model such that $|U| > |M|$. Let $f : N \to M$ be a morphism, where $|f| < |N| \leq \lambda$. We claim that f can be extended to an embedding. Let M' be a structure of cardinality $\leq \lambda$ such that $\operatorname{rng} f \subseteq M' \preceq M$. As T_{rich} is a complete theory, $U \equiv M'$ and, by λ–saturation, there is an elementary embedding $g : M' \to U$. By λ–richness, there is a morphism $h : N \to U$ that extends $gf : N \to U$. As M is λ–saturated, there is an elementary embedding $k : h[N] \to M$. Then $k : U \to M$ is a morphism, so $kh : N \to M$ is the required embedding.

Finally, the implication $3 \Rightarrow 1$ is clear. □

An analogous theorem holds for saturated rich models. The proof is similar.

4.2 THEOREM. Assume that $\mathcal{K}$ is connected. The following are equivalent:
1. some rich model is saturated;
2. all rich models are saturated; and
3. every saturated model $M \vDash T_{\rm rich}$ is rich. □

Observe that when $\mathcal{K}$ is not connected these results hold within each connected component.

Finally, the following theorem shows that saturation and λ–saturation are both equivalent to a certain property of morphisms.

4.3 THEOREM. Let λ be any infinite cardinal. The following are equivalent:
1. all λ–rich models are λ–saturated;
2. all rich models are saturated;
3. if U is rich, $M \equiv U$, and $M \leq U$, then $M \preceq U$;
4. if U is rich, $M \equiv U$, then any morphism $f : M \to U$ is elementary.

PROOF. The equivalence $3 \Leftrightarrow 4$ is clear. We prove $1 \Rightarrow 3$. Suppose that U is rich. We may assume that $\lambda \leq |U|$ (otherwise we prove the claim for a sufficiently large rich model in the same connected component as U; then 3 follows easily). By 1, U is saturated. Let $A \subseteq M$ be any finite set and let M' be a countable model such that $A \subseteq M' \preceq M$. If we show that $M' \preceq U$, $M \preceq U$ follows from the arbitrarity of A. As $M' \equiv U$, by saturation there is a model $M'' \preceq U$ which is isomorphic to M'. Let $f : M' \to M''$ be this isomorphism. Then $f : U \to U$ is a morphism and, as U is rich, an elementary map by 3.5. So $M' \preceq U$ as required. The implication $2 \Rightarrow 3$ is similar.

Finally, we assume 4 and prove that if U is λ–rich then it is λ–saturated. As λ is arbitrary, both $4 \Rightarrow 1$ and $4 \Rightarrow 2$ follow. Let $p(x)$ be a type over some set $A \subseteq U$ of cardinality $< \lambda$. Fix some model $M \equiv_A U$ of cardinality $\leq \lambda$ that realizes $p(x)$. By 4, there is an elementary embedding $f : M \to U$ over A. Hence U realizes $p(x)$. □

4.4 COROLLARY. Let U be a rich model and suppose it is saturated. Then for any $M \equiv N \equiv U$, every morphism $f : M \to N$ is elementary.

PROOF. Let V be a rich model and let $h : N \to V$ be an embedding. As V and U are in the same connected component, they are elementarily equivalent. Then $h : N \to V$ and $hf : M \to V$ are elementary by Theorem 4.3. Then $f : M \to N$ is elementary as well. □

The models M and N in Theorem 4.3 and its corollaries are required to be elementarily equivalent to some rich model. It would be convenient to replace this condition by $M, N \vDash T_{\text{rich}}$ but this is not possible in general: the following example shows that there may be models where T_{rich} holds which are not elementarily equivalent to any rich model.

4.5 EXAMPLE. The language L_0 contains a binary predicate r and the constants c_n, for $n \leq \omega$. The models of $\mathcal{K}$ are the structures of signature L_0 where the following axioms hold:

0. $c_i \neq c_j$ for every distinct $i, j \leq \omega$,
1. $\forall x \ \neg r(x,x)$,
2. $\forall x\, y\ [r(x,y) \leftrightarrow r(y,x)]$,
3. $\exists x\, r(c_i, x) \ \rightarrow\ \neg\exists x\, r(c_j, x)$ for every distinct $i, j \leq \omega$.

So, models are graphs with countably many vertices named; these vertexes are, with one possible exception, isolated. Now for each $n < \omega$ we define $\mathcal{K}_n$ to be the class of models where one of the following conditions holds:

a. $\exists x\ r(c_n, x)$, or

b. $\neg\exists x\ r(c_i, x)$ for every $i \leq \omega$ and there are exactly n triangles (i.e. cliques of size 3).

Finally, let $\mathcal{K}_\omega$ be the class of the models where one of the following conditions holds:

a′. $\exists x\ r(c_\omega, x)$, or

b′. $\neg\exists x\ r(c_k, x)$ and there are more than k triangles for every $k < \omega$

So, $\mathcal{K}_n$, for any $n \leq \omega$, contains two sorts of graphs: those where c_n is the unique constant which is non-isolated and those where all constants are isolated. When all constants are isolated, the graph contains exactly n triangles if $n < \omega$, or infinitely many if $n = \omega$.

As morphisms of $\mathcal{K}$ we take all partial isomorphisms between models in the same component $\mathcal{K}_n$. We shall verify that $\mathcal{K}$ is an inductive amalgamation class. The only axioms that require a proof are K3 and AP. To prove K3 it is enough to observe that models in different components are not elementarily equivalent. To prove AP it suffices to show that if M_1 and M_2 are models in the same component $\mathcal{K}_n$ and $M_1 \cap M_2$ is a common substructure, then there is a model N that is a superstructure of both M_1 and M_2. There are two cases. If $M_i \vDash \exists x\, r(c_n, x)$ for either one of $i \in \{1,2\}$, we let N be the free amalgam of M_1 and M_2 over $M_1 \cap M_2$, that is, $N = M_1 \cap M_2$ with no extra edges added. Otherwise we take $N = M_1 \cup M_2 \cup \{a\}$, were a is a new vertex and let $r^N := r^{M_1} \cup r^{M_2} \cup \{\langle c_n, a\rangle, \langle a, c_n\rangle\}$. This ensures that $N \vDash \exists x\, r(c_n, x)$, so that axiom 3 holds. Axioms 0–2 are clear.

We now describe a countable rich model $U \in \mathcal{K}_n$. This is the disjoint union of two structures $U_{\rm rand}$ and $U_{\rm isol}$: the first is a random graph, and the second contains only isolated vertices. The structure $U_{\rm rand}$ contains c_n, while $U_{\rm isol}$ contains all other constants and infinitely many other vertices. In $U_{\rm rand}$ the vertex c_n is non isolated (no vertex is isolated in a random graph) so axiom 3 holds.

We check that the model U described above is rich. Let $f : M \to U$ be a morphism, i.e. a partial isomorphism, with $|f| < |M| \leq |U|$. We can extend f to f' so that $\{c_i : i \leq \omega\} \subseteq \operatorname{dom} f'$. Let $f' = f_{\rm rand} \cup f_{\rm isol}$ where $\operatorname{rng} f_{\rm rand} \subseteq U_{\rm rand}$ and $\operatorname{rng} f_{\rm isol} \subseteq U_{\rm isol}$. We can extend $f_{\rm rand}$ to an embedding of $M \smallsetminus \operatorname{dom} f_{\rm isol}$ into $U_{\rm rand}$, because this is a random graph. This proves that U is rich.

Consider a structure M which is the disjoint union of a countable random graph and a set of isolated vertices containing all the constants and infinitely many other elements. Since in M all constants are isolated, M is not elementary equivalent to any rich model. But every formula φ true in M also holds in some rich model U (e.g. if c_n does not occur in φ, then φ will hold in $U \in \mathcal{K}_n$). □

The example above motivates the following definition.

4.6 Definition. We say that an inductive amalgamation class is **full** if $M \equiv N$ for any two models $M, N \vDash T_{\rm rich}$ in the same connected component. □

Connected amalgamation classes are trivially full. To sum up we state the following theorem—a corollary of the definition above—which generalizes Theorems 4.1 and 4.3 to the case where $\mathcal{K}$ is full.

4.7 Theorem. Suppose $\mathcal{K}$ is full. Then the following are equivalent:
1. all rich models are saturated;
2. all λ–rich models are λ–saturated;
3. all saturated model $M \vDash T_{\rm rich}$ are rich;,
4. all morphisms between models $M, N \vDash T_{\rm rich}$ are elementary;
5. $M \leq N \;\Leftrightarrow\; M \preceq N$, for any pair of models $M, N \vDash T_{\rm rich}$. □

We also have the following.

4.8 Theorem. The following are equivalent:
1. all rich models are saturated and $\mathcal{K}$ is full;
2. $M \leq N \;\Leftrightarrow\; M \preceq N$, for any pair of models $M, N \vDash T_{\rm rich}$.

PROOF. The direction 1 $\Rightarrow$ 2 is part of Theorem 4.7. For the converse observe that every model M is embeddable in some rich model U, therefore 2 implies that $M \equiv U$. Hence T is full and 1 follows by Theorem 4.7. □

5 Model companions

In this section we generalize some results of [ChaPi], namely Section 3.4 and Proposition 3.5: we show that their results for model companions of models with an automorphism hold in the context of inductive amalgamation classes. We also prove that the existence of model companions is equivalent to fullness of the class plus saturation of rich models.

We fix an inductive amalgamation class $\mathcal{K}$. First, we re-examine the results of the previous section under the following assumption:

\# if $M, N \models T_{\text{rich}}$, then $M \subseteq N \;\Leftrightarrow\; M \leq N$.

This condition is equivalent to claiming that partial isomorphisms are embeddings (i.e. total morphisms) as soon as they are total maps.

5.1 THEOREM. Assume that # holds in $\mathcal{K}$. Then the following are equivalent:
1. all rich models are saturated and $\mathcal{K}$ is full;
2. T_{rich} is model-complete.

PROOF. The model-completeness of T_{rich} is equivalent to

† if $M, N \models T_{\text{rich}}$, then $M \subseteq N \;\Leftrightarrow\; M \preceq N$.

By # we can replace '$\subseteq$' with '$\leq$' in †. Then the equivalence follows from Theorem 4.8. □

We say that $\mathcal{K}$ is **axiomatizable** if there is a theory T_0 such that M is a model if and only if $M \models T_0$. In this case, we also say that $\mathcal{K}$ **is axiomatised by** T_0.

5.2 THEOREM. Assume that $\mathcal{K}$ is axiomatised by a theory T_0. Then $T_{0,\forall} = T_{\text{rich},\forall}$.

PROOF. Every structure modelling T_0 is a model, hence the substructure of a rich model. So $T_{\text{rich},\forall} \subseteq T_{0,\forall}$. Vice versa, as T_0 holds in every model, it holds a fortiori in every rich model, so $T_0 \subseteq T_{\text{rich}}$. □

5.3 THEOREM. Assume $\mathcal{K}$ is axiomatizable by T_0 and assume that # holds in $\mathcal{K}$. Then the following are equivalent:

1. T_{rich} is model-complete.
2. T_{rich} is the model companion of T_0.
3. all rich models are saturated and $\mathcal{K}$ is full,

Vice versa, if T_0 has a model companion, then T_{rich} is this model companion.

PROOF. The equivalences $1 \Leftrightarrow 2 \Leftrightarrow 3$ are clear by Theorems 5.1 and 5.2. To prove the second claim, we assume T_0 has a model companion T_c. To see that $T_c \subseteq T_{\text{rich}}$ it suffices to observe that, by #, rich models are existentially closed, so T_c holds in every rich model. To prove the converse inclusion, let $M_0 \vDash T_c$ be any structure. We claim that $M_0 \vDash T_{\text{rich}}$. As $T_{0,\forall} = T_{0,\forall}$, every structure $M \vDash T_c$ is a substructure of a rich model. Viceversa, every rich model is a substructure of some $M \vDash T_c$. So we can construct a chain of substructures

$$M_0 \subseteq U_0 \subseteq M_1 \subseteq U_1 \subseteq M_2 \subseteq \ldots\ldots,$$

where $M_i \vDash T_c$ and U_i is a rich model. It follows that $M_i \preceq M_{i+1}$ and $U_i \preceq U_{i+1}$. Let

$$U_\omega \ := \ \bigcup_{i\in\omega} U_i \ = \ \bigcup_{i\in\omega} M_i.$$

Then $M_0 \preceq U_\omega$. The union of a chain of rich models is ω–rich, so the theorem follows. □

5.4 REMARK. The requirement of fullness in 3 of Theorem 5.3 is necessary. The rich models in Example 4.5 are all saturated, but T_{rich} is not model-complete: the formula $\exists y\, r(x,y)$ is not equivalent over T_{rich} to any universal formula. In fact $\exists y\, r(x,y)$ is not preserved under substructure: if U is a rich model in $\mathcal{K}_\omega$, then $U \vDash \exists y\, r(c_\omega, y)$ but in the model $M \subseteq U$ constructed at the end of Example 4.5, we have $\neg\exists y\, r(c_\omega, y)$. □

6 Baire categories of first-order expansions

Let ***T*** be a complete theory in the language ***L***. Let $\boldsymbol{L_0}$ be the language L enriched with finitely many new relation and function symbols.

6.1 NOTATION. From now on, we shall denote a structure of signature L_0 by a pair (N,σ), where N is a structure of signature L and σ is the interpretation of the symbols in $L \smallsetminus L_0$. So the results of the previous sections will hold for (M,σ), (N,τ) etc. in place of M, N, etc. Let T_0 be any theory of signature L_0 containing T. We write:

$$\mathbf{Exp}(\boldsymbol{N},\boldsymbol{T_0}) \ := \ \Big\{\sigma \ : \ (N,\sigma) \vDash T_0\Big\},$$

We write **Exp(*N*)** for $\mathrm{Exp}(N, T)$. □

There is a canonical topology on $\mathrm{Exp}(N)$, cf. [Ivan]. For a sentence φ with parameters in N we define $[\varphi]_N := \{\sigma : (N, \sigma) \models \varphi\}$. The topology on $\mathrm{Exp}(N)$ is generated by the open sets of the form $[\varphi]_N$ with φ quantifier-free. When N is countable, this topology is completely metrizable: fix an enumeration $\{a_i : i \in \omega\}$ of N, define $d(\sigma, \tau) = 2^{-n}$, where n is the largest natural number such that for every tuple a in $\{a_0, \ldots, a_{n-1}\}$ and any symbols r, f in $L_0 \smallsetminus L$,

$$a \in r^\sigma \Leftrightarrow a \in r^\tau \quad \text{and} \quad f^\sigma(a) = f^\tau(a),$$

where r^σ is short for $r^{(N,\sigma)}$. When such an n does not exist, $d(\sigma, \tau) = 0$.

The reader may easily verify that this metric is complete. We shall check that it induces the topology defined above. Fix n and τ. Let φ be the conjunction of the formulas of the form $fa = b$ and ra which hold in (N, τ) for some $b \in N$ and some tuple a from $\{a_0, \ldots, a_n\}$. Then

$$[\varphi]_N \;\subseteq\; \{\sigma : d(\sigma, \tau) < 2^{-n}\}$$

Vice versa, let φ be a sentence with parameters in N, and take an arbitrary $\tau \in [\varphi]_N$. Let A be the set of parameters occurring in φ. Let n be large enough that

$$\{t^\tau(a) : a \subseteq A \text{ and } t \text{ is a subterm of a term appearing in } \varphi\} \;\subseteq\; \{a_0, \ldots, a_{n-1}\}.$$

Clearly $(N, \sigma) \models \varphi$ for any σ at distance $< 2^{-n}$ from τ so

$$\{\sigma : d(\sigma, \tau) < 2^{-n}\} \subseteq \; [\varphi]_N$$

as required.

If $g : M \to N$ is an isomorphism and $\sigma \in \mathrm{Exp}(M)$ we write $\boldsymbol{\sigma^g}$ for the unique expansion of N that makes $g : (M, \sigma) \to (N, \sigma^g)$ an isomorphism. Explicitly, for every predicate r, every function f in $L_0 \smallsetminus L$, and every tuple $a \in N$,

$$(N, \sigma^g) \models r\,a \quad \Longleftrightarrow \quad (M, \sigma) \models r\,g^{-1}a$$

$$(N, \sigma^g) \models f\,a = b \quad \Longleftrightarrow \quad (M, \sigma) \models g\,f\,g^{-1}a = b$$

When $M = N$ and g is an automorphism, we say that σ is **conjugate** to σ^g by g.

We write $T_{0,\forall}$ for the set of consequences of T_0 that are universal modulo T. Note that $\mathrm{Exp}(N, T_0) \subseteq \mathrm{Exp}(N, T_{0,\forall}) \subseteq \mathrm{Exp}(N)$.

6.2 Lemma. Let T have a countable saturated model N, and consider an arbitrary expansion T_0 of T to the signature L_0. Then $\mathrm{Exp}(N, T_{0,\forall})$ is the closure of $\mathrm{Exp}(N, T_0)$ in the above topology.

PROOF. Let $\tau \in \text{Exp}(N, T_{0,\forall})$ and suppose $\tau \in [\varphi]_N$. We claim that τ is adherent to $\text{Exp}(N, T_0)$. As (N, τ) models the universal consequences of T_0, there exists some $(N', \tau') \vDash T_0$ such that $(N, \tau) \subseteq (N', \tau')$. Let $A \subseteq N$ be the set of parameters occurring in φ. As we can assume that N' is countable and saturated (in L), it is isomorphic to N over A, hence $[\varphi]_N$ contains some element of $\text{Exp}(N, T_0)$.

Vice versa, suppose that $\tau \notin \text{Exp}(N, T_{0,\forall})$. Then for some parameter- and quantifier-free formula $\varphi(x)$ we have $T_0 \vdash \forall x\, \varphi(x)$ and $(N, \tau) \vDash \neg\varphi(a)$. Then the open set $[\neg\varphi(a)]_N$ separates τ from $\text{Exp}(N, T_0)$. □

6.3 ASSUMPTIONS. In what follows, we work with a complete theory $\boldsymbol{T}$ with quantifier elimination in the language $\boldsymbol{L}$. We require that T is small and indicate with $\boldsymbol{N}$ some fixed countable saturated model of T. Let $\boldsymbol{T_0}$ be an expansion of T of signature $\boldsymbol{L_0}$, and assume that $\boldsymbol{T_0}$ is universal modulo T. Let $\boldsymbol{\mathcal{K}}$ be an inductive amalgamation class axiomatized by T_0. We assume that hypothesis # of Section 5 holds in $\mathcal{K}$. □

6.4 DEFINITION. We say that $\pi(x)$ is a **quantifier-free quasifinite type** if it is quantifier-free and contains only finitely many formulas not in L. We say that (M, σ) is a **smooth model** if it realizes every quantifier-free quasifinite type which has finitely many parameters and is finitely consistent in (M, σ). When (N, σ) is a smooth model we simply say that σ is a **smooth expansion**. □

We write $\boldsymbol{p_{\restriction L}(x)}$ for the reduct to L of the type $p(x)$. Since T eliminates quantifiers, when convenient we may assume that $p_{\restriction L}(x)$ is quantifier-free. So we may assume that quantifier-free quasifinite types have the form $p_{\restriction L}(x) + \varphi(x)$ for some quantifier-free formula $\varphi(x)$. When T is ω-categorical, any expansion is smooth.

We say that a model (M, σ) is **existentially closed** if every existential M–sentence that holds in some model (U, υ) such that $(M, \sigma) \subseteq (U, \upsilon)$ holds in (M, σ) as well. We opt for this definition though in our context it would be more natural to replace $\subseteq$ with $\leq$. The proofs in Section 7 are more direct if we use the stronger notion of existential closure given here.

By Lemma 6.2 and Assumptions 6.3, $\text{Exp}(N, T_0)$ is a closed subset of $\text{Exp}(N)$, hence it is complete.

6.5 FACT. The set of existentially closed smooth expansions is comeagre in $\text{Exp}(N, T_0)$.

PROOF. Observe that existential closure and smoothness can be obtained simultaneously if in Definition 6.4 we replace 'finitely consistent in (M, σ)' with 'finitely consistent in some model which is a superstructure of (M, σ)'.

Let $\pi(x)$ be a quantifier-free quasifinite type with finitely many parameters in N.

Suppose $\pi(x)$ is finitely consistent in some model (V, υ), where $N \subseteq V$. We prove that the expansions of N that realize $\pi(x)$ are an open dense set. As T is small, there are at most countably many such types $\pi(x)$, so the fact follows.

Let φ be an existential N–sentence consistent with T_0. We may assume that $(V, \upsilon) \vDash \varphi + \exists x\, \pi(x)$. Let $(M, \sigma) \preceq (V, \upsilon)$ be such that M is countable and saturated, $N \subseteq M$, and M contains a realization of $\pi(x)$. As M and N are both countable and saturated, there is an isomorphism $g : M \to N$ that fixes the parameters of $\pi(x)$. Then $(N, \sigma^g) \vDash \varphi + \exists x\, \pi(x)$, so σ^g is the required expansion. □

6.6 Fact. Let $T_{\text{rich},\forall\exists}$ be the set of $\forall\exists$–consequences of T_{rich}. Then $T_{\text{rich},\forall\exists}$ holds in every existentially closed model.

Proof. If (M, σ) is existentially closed and (V, υ) is a model extending (M, σ) then $(M, \sigma) \preceq_1 (V, \upsilon)$. As we can always take (V, υ) rich, $(M, \sigma) \vDash T_{\text{rich},\forall\exists}$. □

6.7 Corollary. The set of expansions that model $T_{\text{rich},\forall\exists}$ is a comeagre subset of $\text{Exp}(N, T_0)$. □

6.8 Example. Let T be any complete small theory with quantifier elimination in the language L. Let $L_0 \smallsetminus L$ contain only a unary relation symbol r and let $T_0 = T$. We define an inductive amalgamation class $\mathcal{K}$. The models of $\mathcal{K}$ are the structures that model T_0. The morphisms of $\mathcal{K}$ are partial isomorphisms that have a domain of definition which is algebraically closed in T, as well as any restriction of these maps. It is easy to verify that all the axioms of Section 2 hold in $\mathcal{K}$ (free amalgamation suffices to prove AP). Hypothesis # of Section 5 is trivially satisfied and all the conditions in 6.3 are met.

Let $\text{acl}(A)$ denote the algebraic closure in T. If $\text{acl}(\varnothing) \neq \varnothing$ the class is not connected: the set $\{a \in \text{acl}(\varnothing) : (M, \sigma) \vDash r(a)\}$ determines the connected component of the model (M, σ). In [ChaPi] it is proved that if T eliminates the $\exists^\infty$ quantifier, then T_0 has a model companion: T_{rich}. In this case, T_{rich} is $\forall\exists$–axiomatizable, so, by Corollary 6.7, we have that $\text{Exp}(N, T_{\text{rich}})$ is comeagre.

Finally, an example of an expansion that is *not* smooth. Suppose T is the theory of the algebraically closed field of some fixed characteristic and let N be an algebraically closed fields of infinite transcendence degree. The expansion where $r(x)$ holds exactly for the elements of $\text{acl}(\varnothing)$ is not smooth. □

6.9 Example. Let T and L be as in Example 6.8. Let $L_0 \smallsetminus L$ contain two unary function symbols f and f^{-1} and let T_0 be the theory that says that f is an automorphism with inverse f^{-1}. We need a symbol for the inverse of f because we want T_0 to be universal. The class $\mathcal{K}$ is defined as in Example 6.8. Here the amalgamation property is not trivial: when it holds one says that T has the PAPA [Lasc2]. So suppose T has the PAPA. Then hypothesis # of Section 5 is again trivially satisfied and all the conditions in 6.3 are met.

This class is not connected. As in the examples above, the restriction of f to $\mathrm{acl}(\emptyset)$ determines the connected component of $\mathcal{K}$ to which the model belongs. It is considerably more difficult to find a condition which guarantees the model-completeness of T_{rich} [BaShe]. An important example is the case where T is the theory of algebraically closed fields [ChaHr]. Then T_{rich} is also kown as ACFA. Let N be a countable algebraically closed field of infinite transcendence degree. By Corollary 6.7 we may conclude that $\mathrm{Exp}(N, T_{\mathrm{rich}})$ is comeagre.

7 Truss-generic expansions

Throughout this section we shall adopt the notation and the assumptions stated in 6.3. In particular, N will be a given countable saturated model of T. Moreover, we shall assume that $\mathcal{K}$ is connected, so that T_{rich} is a complete theory. We shall write Y for the set of existentially closed smooth expansions of N. From Fact 6.5 we know that Y is a comeagre subset of $\mathrm{Exp}(N, T_0)$. We may consider Y as a topological space in its own right with the topology inherited from $\mathrm{Exp}(N, T_0)$.

If $\pi(x, y)$ is a quantifier-free quasifinite type then in every smooth model the infinitary formula $\exists y\, \pi(x, y)$ is equivalent to a type, namely the type $\{\exists y\, \varphi(x, y) : \varphi(x, y) \in \pi(x, y)\}$. Types of this form are called **existential quasifinite**.

Let b be a finite tuple in N. For any $\alpha \in Y$ we define the **1-diagram** of α at b as

$$\mathbf{dg}_{\restriction 1}(\boldsymbol{\alpha}, \boldsymbol{b}) \quad := \quad \big\{\varphi(b) \;:\; \varphi(x) \text{ universal or existential and } (N, \alpha) \vDash \varphi(b)\big\},$$

and write $\boldsymbol{D_b}$ for the set of 1-diagrams at b. On D_b we define a topology whose basic open sets are of the form

$$[\,\boldsymbol{\pi(b)}\,]_{\boldsymbol{D}} \quad = \quad \big\{\mathrm{dg}_{\restriction 1}(\alpha, b) \;:\; (N, \alpha) \vDash \pi(b)\big\},$$

where $\pi(x)$ is any existential quasifinite type. We say that b is an **e-isolated tuple** in α if $\mathrm{dg}_{\restriction 1}(\alpha, b)$ is an isolated point of D_b. We may say b is e-isolated **by** $\pi(x)$ in α.

It is sometimes convenient to use the syntactic counterpart of D_b which we now define. If $p(x)$ is a complete type, we write $\boldsymbol{p_{\restriction\forall}(x)}$, respectively $\boldsymbol{p_{\restriction\exists}(x)}$ for the set of universal, respectively existential, formulas in $p(x)$. We write $\boldsymbol{p_{\restriction 1}(x)}$ for $p_{\restriction\forall}(x) + p_{\restriction\exists}(x)$. We say that $p(x)$ is realized in Y if it is realized in some (N, σ) for some $\sigma \in Y$. Let $\boldsymbol{S_x}$ be the set of types of the form $p_{\restriction 1}(x)$, where $p(x)$ is some complete parameter-free type realized in Y. On S_x define the topology where the basic open sets are of the form

$$[\,\boldsymbol{\pi(x)}\,]_{\boldsymbol{S}} \quad = \quad \Big\{ q_{\restriction 1}(x) \;:\; \pi(x) \;\subseteq\; q(x) \Big\},$$

where $\pi(x)$ is some existential quasifinite type, and $q(x)$ ranges over the parameter-free types realized in Y. When $[\pi(x)]_S$ isolates $p_{\restriction 1}(x)$ in S_x, we say that $p(x)$ is **e-isolated by** $\pi(x)$.

7.1 FACT. Let b be a tuple in N and let $p_{\restriction L}(x)$ be the parameter-free type of b in the language L. There is a homeomorphism $h : D_b \;\rightarrow\; [p_{\restriction L}(x)]_S$. For every existential quasifinte type $\pi(x)$ containing $p_{\restriction L}(x)$, the image under h of the set $[\pi(b)]_D$ is the set $[\pi(x)]_S$.

PROOF. Let h be the function that takes the universal diagram $\mathrm{dg}_{\restriction 1}(\alpha, b)$ to the universal type $\{\varphi(x) \,:\, \varphi(b) \in \mathrm{dg}_{\restriction 1}(\alpha, b)\}$.

It is clear that h maps D_b injectively to S_x. For surjecivity, let $q(x)$ be a complete parameter-free type realized in Y, say $(N, \sigma) \vDash q(a)$ for some $\sigma \in Y$, and suppose that $q_{\restriction 1}(x)$ belongs to $[\pi(x)]_S$. As $p_{\restriction L}(x) \subseteq q(x)$, there is an isomorphism $g : N \rightarrow N$ such that $g(a) = b$. Then $q_{\restriction 1}(x)$ is the image of $\mathrm{dg}_{\restriction 1}(\sigma^g, b)$ under h. This proves surjectivity. □

From this fact it is clear that b is e-isolated in α if and only if $p(x)$, the parameter-free type of b in (N, α), is e-isolated. The following lemma is also clear.

7.2 LEMMA. Let $p(x)$ be a complete parameter-free type realized in Y and let $\pi(x)$ be an existential quasifinite type such that $p_{\restriction L} \subseteq \pi(x) \subseteq p(x)$. Then the following are equivalent

1. $p(x)$ is e-isolated by $\pi(x)$;
2. $\pi(x) \;\rightarrow\; p_{\restriction 1}(x)$ holds in every $\sigma \in Y$. □

7.3 DEFINITION. Let $\alpha \in Y$ and let b be a finite tuple in N. We say that (N, α) is an e-atomic model, or, for short, that α is **e-atomic**, if every finite tuple in N is e-isolated. In other words, (N, α) realizes $p_{\restriction 1}(x)$ only if $p(x)$ is e-isolated. □

The notion of e-atomic is virtually the same as Ivanov's notion of $(A, \exists)$–atomic in [IVAN] but, since the context is different, a circumstantial comparison is not straightforward.

7.4 REMARK. As remarked in Section 6, when T is ω–categorical, every expansion is smooth. When T_{rich} is model-complete, every model of T_{rich} is existentially closed and every formula is equivalent to an existential (or, equivalently, to a universal) formula. So, if T_{rich} is model-complete and T is ω–categorical the e-atomic expansions are exactly those such that (N, α) is an atomic model of T_{rich}. □

7.5 THEOREM. Any two e-atomic expansions are conjugated.

PROOF. Let α and β be e-atomic. We prove the following claim: any finite 1–elementary map $f : (N, \alpha) \to (N, \beta)$ can be extended to an isomorphisms. By Fact 6.6 atomic expansion model $T_{\text{rich},\forall\exists}$. Recall that we are working in a connected class, so T_{rich} is a complete theory and any two models of $T_{\text{rich},\forall\exists}$ are 1–elementarily equivalent. In other words, the empty map $\varnothing : (N, \alpha) \to (N, \beta)$ is 1–elementary, so the theorem follows from the claim.

To prove the claim it suffices to show that for any finite tuple b we can extend f to some 1–elementary map defined on b. The claim then follows by back and forth. Let a be an enumeration of dom f. The tuple $a\,b$ is e-isolated in α, say by some existential quasifinite type $\pi(v, x)$. Let $p(v, x) = \text{tp}(a, b)$. By fattening π if necessary, we may assume that it contains $p_{\restriction L}(v, x)$. Since β is smooth and f is 1–elementary, the type $\pi(fa, x)$ is realized in β, say by c. By lemma 7.2, $\pi(v, x) \ \to p_{\restriction 1}(v, x)$ holds both in α and β, so $f \cup \{\langle b, c\rangle\}$ gives the required extension. □

7.6 FACT. If an e-atomic expansion exists, then the set of e-atomic expansions is comeagre in $\text{Exp}(N, T_0)$.

PROOF. We prove that the set of e-atomic expansions is a dense G_δ subset of Y, hence comeagre in $\text{Exp}(N, T_0)$.

To prove density, let $\psi(x)$ be a parameter-free existential formula. Let $a \in N$ be such that $\psi(a)$ is consistent with T_0. We show that $(N, \alpha) \vDash \psi(a)$ for some e-atomic α. Write $p_{\restriction L}(x)$ for the parameter-free type of a in the signature L. Let β be any e-atomic expansion and let c be a realization of $p_{\restriction L}(x) \wedge \psi(x)$ in (N, β). Let g be an automorphism of N such that $g(c) = a$. Then $\alpha := \beta^g$ is the required expansion. Hence the set of e-atomic expansions is dense.

We now prove that the set of e-atomic expansions is a G_δ subset of Y. Let b be a finite tuple and denote by X_b the set of expansions in Y where b is e-isolated. It suffices to prove that X_b is an open subset of Y.

Let $\alpha \in X_b$ and let $[\pi_\alpha(b)]_D$ be the basic open subset of D_b that isolates $\text{dg}_{\restriction 1}(\alpha, b)$. We may assume $\pi_\alpha(b)$ has the form $\exists y\, [\, p_{\alpha\restriction L}(b, y) \wedge \varphi_\alpha(b, y)\,]$. So let a_α be a witness of the existential quantifier. We have that $Y \cap [\varphi_\alpha(a_\alpha, b)]_N \subseteq X_b$. It follows that

$$Y \ \cap \bigcup_{\alpha \in X_b} [\varphi_\alpha(a_\alpha, b)]_N \ = \ X_b.$$

Hence X_b is an open subset of Y. □

In [Tru1], a notion of generic automorphisms is introduced and a number of examples are given of countable, ω–categorical structures that have generic automorphisms. The following definition, which appears in [Ivan], generalizes the notion of generic automorphisms to arbitrary expansions.

7.7 Definition. We say that an expansion τ is **Truss-generic** if $\{\tau^g : g \in \mathrm{Aut}(N)\}$ is a comeagre subset of $\mathrm{Exp}(N, T_0)$.

7.8 Remark. There is at most one comeagre subset of $\mathrm{Exp}(N, T_0)$ of the form $\{\tau^g : g \in \mathrm{Aut}(N)\}$. This is because any two sets of this form are either equal or disjoint, and two comeagre sets in a Baire space have nonempty intersection.

The following theorem answers a question of [Tru2].

7.9 Theorem. Let α be any expansion. Then the following are equivalent:
1. α is e-atomic;
2. α is Truss-generic.

Proof. Let α be e-atomic. By Fact 7.6, the set X of e-atomic expansions is comeagre. By Corollary 7.5, and because X is closed under conjugacy by elements of $\mathrm{Aut}(N)$, X is of the form $\{\tau^g : g \in \mathrm{Aut}(N)\}$ for any e-atomic τ. By Remark 7.8, X is exactly the set of Truss-generic expansions.

Vice versa, let α be Truss-generic. As smoothness and existential closure are guaranted by Fact 6.5, we only need to prove that α omits $p_{\restriction 1}(x)$ for any complete parameter-free type $p(x)$ that is not e-isolated. It suffices to prove that the set of expansions in Y that omit $p_{\restriction 1}(x)$ is dense G_δ in Y, hence comeagre in $\mathrm{Exp}(N, T_0)$. Then some Truss-generic expansion omits it and, as Truss-generic expansions are conjugated, the same holds for α.

Denote by X_b the set of expansions in Y that model $\neg p_{\restriction 1}(b)$. The set of expansions in Y that omit $p_{\restriction 1}(x)$ is the intersection of X_b as the tuple b ranges over N. So, if we can show that X_b is open dense in Y, we are done.

First we prove density. Let $\psi(a, b)$ be an existential formula where a and b are disjoint. We need to show that there is an expansion in Y that models $\psi(a, b) \wedge \neg p_{\restriction 1}(b)$. Let $q_{\restriction L}(z, x)$ be the parameter-free type of a, b in the language L. We claim that $\psi(z,x)\wedge q_{\restriction L}(z,x)\wedge\neg p_{\restriction 1}(x)$ is consistent in Y, say it is realized by a', b' in some expansion $\sigma\in Y$. If not, then $\psi(z,x)\wedge q_{\restriction L}(z,x)\ \rightarrow\ p_{\restriction 1}(x)$ holds in every expansion in Y, which contradicts that $p(x)$ is not e-isolated and proves the claim. There is an automorfism $g: N\rightarrow N$ such that $g(a'\,b') = a\,b$. We conclude that $\psi(a,b)\wedge\neg p_{\restriction 1}(b)$ holds in (N,σ^g).

Now we prove that X_b is open in Y. Let $\sigma\in X_b$. We shall show that σ belongs to a basic open set contained in X_b. If $(N,\sigma)\vDash\neg p_{\restriction\forall}(b)$ the claim is obvious, so suppose that $(N,\sigma)\vDash\neg\varphi(b)$ for some existential formula $\varphi(x)$. The expansions in Y are existentially closed, hence (see, for instance, Theorem 7.2.4 in [Hodg]) there is an existential formula $\psi(x)$, consistent in (N,σ), such that $\psi(x)\rightarrow\varphi(x)$ holds for every $\tau\in Y$. Then $\sigma\in[\exists x\,\psi(x)]_N\subseteq X_b$ as required. □

7.10 Lemma. The following are equivalent:
1. Truss-generic expansions exist;
2. for every finite b, the isolated points are dense in D_b.
3. for every finite x, the isolated points are dense in S_x.

Proof. The equivalence $2\Leftrightarrow 3$ is clear by Fact 7.1. The direction $1\Rightarrow 3$ is a corollary of Theorem 7.9. To prove the converse we assume 2 and construct a set Δ which is the existential diagram of an e-atomic model.

The diagram Δ is defined by finite approximations. Assume that at stage i we have a finite set Δ_i of existential N–sentences that is consistent with T_0. Then we define Δ_{i+1} as follows:

If i is even, let b a tuple that enumerates all paramenters in Δ_i. Let α be such that $\mathrm{dg}_{\restriction 1}(\alpha,b)$ is isolated in D_b, say by the type $\exists y\,[p_{\restriction L}(b,y)\wedge\varphi(b,y)]$ where $\varphi(b,y)$ is quantifier-free. Let a be such that $p_{\restriction L}(b,a)\wedge\varphi(b,a)$ and define $\Delta_{i+1}:=\Delta_i+\varphi(b,a)$. If i is odd consider the least type of the form $p_{\restriction L}(x)+\varphi(x)$, where $\varphi(x)$ is quantifier-free, that is consistent with $T_0+\Delta_i$ and has not been considered yet. Let c be such that $T_0+p_{\restriction L}(c)+\varphi(c)$ holds for some expansion and define $\Delta_{i+1}:=\Delta_i+\varphi(c)$.

Let (N,α) be the model with diagram Δ. It is clear the even stages ensure that every type realized in (N,α) is e-isolated while the odd stages guarantee simultaneously smoothness and existential closure. □

8 Examples

We work under the assumptions and with the notation stated at the beginning of Section 7.

8.1 EXAMPLE. Truss-generic automorphisms of the random graph. Let L be the language of graphs and let T be the theory of the random graph. Let L_0 and T_0 be as in Example 6.9. It is not difficult to verify that the corresponding class $\mathcal{K}$ has the amalgamation property. It is known [KIK] that T_0 has no model companion, hence rich models are not saturated. The existence of Truss-generic automorphisms of the random graph has been first proved in [TRU1] and extended to generic tuples in [HHLS], essentially using [HRU2]. These proofs use amalgamation properties of finite structures.

The proof of the proposition below contains a description of isolated tuples in a well understood case. The existence of Truss-generic automorphisms of the random graph follows by the proposition and Theorem 7.10. This proof is by no means shorter than the proof in [HHLS]. It also uses [HRU2].

PROPOSITION. Let T be the theory of the random graph and let N be a countable random graph. Let L_0 and T_0 be as above (i.e. as in Example 6.9). Then for every finite tuple b in N, the e-isolated points in D_b are dense.

By the main result in [HRU2], for every finite set B of a random graph N there is a finite set A such that $B \subseteq A \subseteq N$ and every partial isomorphism $g : N \to N$ with $\operatorname{dom} g, \operatorname{rng} g \subseteq B$ has an extension to an automorphism of A.

Let $\psi(b)$ be any existential formula consistent with T_0. We shall show that $[\psi(b)]_D$ contains an isolated point. Let (N, α) be a model that realizes $\psi(b)$. We shall show that $[\psi(b)]_D$ contains an isolated point. By the result in [HRU2] mentioned above, there is a model (N, σ) which has a finite substructure $(A, \sigma{\restriction}A)$ that models $\psi(b)$. We may assume that σ is rich. Let $\varphi(a, b)$ be the quantifier-free diagram of A in (N, σ). We claim that $\exists z\, \varphi(z, b)$ isolates a point of D_b, namely $\operatorname{dg}_{\restriction 1}(\sigma, b)$.

To prove the claim, let $\tau \in Y$ model $\exists z\, \varphi(z, b)$ and prove that $(N, \tau) \equiv_{1,b} (N, \sigma)$. As $\varphi(a, b)$ is a diagram of a substructure we can assume (N, τ) and (N, σ) overlap on A so, as they both are existentially closed and can be amalgamated over A, they are 1-elementarily equivalent. □

8.2 EXAMPLE. Cycle-free automorphisms of the random graph. Let L, T, N, and L_0 be as in Example 8.1. The theory T_0, besides of saying that f is an automorphism with inverse f^{-1}, contains the axioms $\forall x\, f^n x \neq x$, for every positive integer n. These axioms claim that f has no finite cycles. It is not difficult to verify that the corresponding class $\mathcal{K}$ has the amalgamation property (the morphisms are all partial isomphisms between models). It is known [KuMac] that T_0 has a model-companion, hence rich models are saturated. Now we prove that there is no Truss-generic.

Suppose for a contradiction that there exists a Truss-generic automorphism τ. Let b be an element of N. As T is ω-categorical, existential quasifinite types are equivalent to existential formulas. So, by Theorem 7.10, there is an existential formula $\varphi(b)$ that isolates $\mathrm{dg}_{\restriction 1}(\tau, b)$ in D_b. As the symbol f^{-1} can be eliminated at the cost of a few extra existential quantifiers, we can assume that it does not occur in $\varphi(b)$. Let n be a positive integer which is larger than the number of occourrences of the symbol f in $\varphi(b)$. Denote by f_τ the interpretation of f in (N, τ). Let $A \subseteq N$ be a finite set containing b and such that the sets $\{c, f_\tau c, \ldots, f_\tau^{n-1}\}$, for $c \in A$, are pairwise disjoint and let B be the union of all these sets. Clearly we can chose A such that B contains all witnesses of the existential quantifiers in $\varphi(b)$. This latter requirement guarantees that if α is an expansion such that $\alpha{\restriction}B = \tau{\restriction}B$, then $(N, \alpha) \vDash \varphi(b)$. Define $d := f_\tau^n b$ and $e := f_\tau d$. Let $e' \in N$ realize the type $\mathrm{tp}_{\restriction L}(e/f_\tau[B])$ and be such that $r(b, e) \not\leftrightarrow r(b, e')$. As $b \notin f_\tau[B]$, the theory of random graph ensure the existence of such e'. Let $g := f_\tau{\restriction}B \cup \{\langle d, e'\rangle\}$. We claim that $g : N \to N$ is a partial isomorphism. To prove the claim it suffices to check that $r(a, d) \leftrightarrow r(ga, e')$ for every $a \in B$. The equivalence holds with e for e'. As $ga \in f_\tau[B]$, by the choice of e' we have $r(ga, e) \leftrightarrow r(ga, e')$. Then $r(a, d) \leftrightarrow r(ga, e')$ follows. Finally, by the homogenety of N we obtain an extension of g to a cycle-free automorphism of N. This yields an expansion α. By construction $\alpha{\restriction}B = \tau{\restriction}B$ so, as observed above, $(N, \alpha) \vDash \varphi(b)$. But (N, τ) and (N, α) disagree on the truth of $r(b, f^{n+1}b)$. This contradicts that $\varphi(b)$ isolates $\mathrm{dg}_{\restriction 1}(\tau, b)$. □

Example 8.2 shows that saturation of rich models alone is not sufficient to guarantee the existence of Truss-generic expansions. The following corollary of Theorem 7.10 gives a sufficient condition.

8.3 Corollary. If a saturated countable rich model exists, then N has a Truss-generic expansion.

Proof. If a saturated rich model exists, then T_{rich} is model-complete by Theorem 5.1. So T_{rich} is the theory of the existentially closed models and every formula is equivalent to an existential (or, equivalently, to a universal) one. Then S_x is the set of all complete parameter-free types consistent with T_{rich}. Though the topology on S_x is not the standard one, the usual argument (e.g. Theorem 4.2.11 of [Mark]) suffices to prove that the isolated types are dense. □

8.4 Example. Poizat's black fields, uncollapsed version. This is a paradigmatic example among many possible versions of Hrushovski's amalgamation constructions. We refer to [Poiz] for all unproved claims. Let L be the language of rings and T the theory of algebracally closed fields of a given characteristic. Let L_0 contain a unary predicate r. Define

$$\delta(A) \quad = \quad 2 \cdot \deg(A) - |r(A)|,$$

where $\deg(A)$ is the trascendence degree of A. Define a universal theory T_0 translating into first-order sentences the requirement that $0 \leq \delta(A)$ holds for every finite set A.

Fix $(M, \sigma) \vDash T_0$ and let $A \subseteq M$. We write **$A \sqsubseteq M$**, if for every finite $B \subseteq M$ we have that $\delta(A \cap B) \leq \delta(B)$. Let **$\operatorname{acl}(A)$** denote the algebraic closure of A in the signature L. Observe that, as T is a complete theory, this does not depend on M nor on σ. We say that A is a **self-sufficient** in M if $\operatorname{acl}(A) = A \sqsubseteq M$. We write **$\operatorname{sscl}(A)$** for the intersection of all self-sufficient subsets of M containing A. This is called the **self-sufficient closure** of A; it clearly depends on σ though we shall not display it in the notation. It is not difficult to prove that $\operatorname{sscl}(A)$ is self-sufficient.

The morphisms of $\mathcal{K}$ are the maps $f : (N, \tau) \to (M, \sigma)$ that have an extension to a partial isomorphim $h : (N, \tau) \to (M, \sigma)$ with $\operatorname{dom} H$ and $\operatorname{rng} H$ self-sufficient in (N, τ) and (M, σ) respectively. It is easy to show that if $(N, \tau) \preceq_1 (M, \sigma) \vDash T_0$, then N is self-sufficient in (M, σ). So axiom K2 holds. Axiom AP is easily verified by free amalgamation and all the other axioms are clear.

This definition of morphism implies that $(N, \tau) \leq (M, \sigma)$ if and only if N is self-sufficient in (M, σ). From the definability of Morley rank in algebraically closed fields, it follows that for any (M, σ), $(N, \tau) \vDash T_{\text{rich}}$ such that $(M, \sigma) \subseteq (N, \tau)$, we have $(M, \sigma) \leq (N, \tau)$. So assumption # of Section 5 holds.

Finally, no element of $\operatorname{acl}(\varnothing)$ satisfies $r(x)$ so the class is connected. Rich models are saturated (this uses the definability of Morley rank in algebraically closed fields). Then by 5.3 above T_{rich} is the companion of T_0. Also, T_{rich} is ω–stable with Morley rank $\omega + \omega$. Therefore, in these examples T_{rich} is small and e-atomic models exist by Corollary 8.3. □

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